The Rohlin Property for $\mathbf{Z}^2$-Actions on UHF Algebras

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June 1996

Abstract

We define a Rohlin property for $\mathbf{Z}^2$-actions on UHF algebras and show a non-commutative Rohlin type theorem. Among those actions with the Rohlin property, we classify product type actions up to outer conjugacy. We consider two classes of UHF algebras. For UHF algebras in one class including the CAR algebra, there is one and only one outer conjugacy class of product type actions and for UHF algebras in the other class, contrary to the case of $\mathbf{Z}$-actions, there are infinitely many outer conjugacy classes of product type actions.

1 Introduction

A non-commtative Rohlin property was introduced by A. Connes for classification of (single) automorphisms of von Neumann algebras ([3, 4]), and this property was generalized for example by A. Ocneanu ([20, 21]) to systems of commuting automorphisms and further to actions of discrete amenable groups. On the other hand, this notion has also proved useful in the framework of C^*-algebras ([1, 6, 7, 12, 13, 15, 16]), and in [15, 16] Kishimoto established a non-commutative Rohlin type theorem for automorphisms of UHF algebras (and some AF algebras) and classified automorphisms (i.e., $\mathbf{Z}$-actions) with the Rohlin property up to outer conjugacy.

The purpose of the paper is to extend Kishimoto's work to $\mathbf{Z}^2$-actions. Motivated by [15, 16], in Section 2 we introduce notions of Rohlin property and uniform outerness to $\mathbf{Z}^N$-actions on unital C^*-algebras. In the UHF algebra case and $N = 1$, the uniform outerness was shown to be the same as the ordinary outerness of the relevant automorphism on the GNS von Neumann algebra obtained via the trace ([15]). Our main theorem here says that for $\mathbf{Z}^2$-actions on UHF algebras the Rohlin property characterizes the uniform outerness. The

main idea of the proof is similar to the one in [16], but to avoid additional technical problems we make use of the stability i.e., the vanishing of 1-cohomology obtained in [12].

In Section 3 we introduce three notions of conjugacy to $\mathbf{Z}^2$-actions, i.e, approximate conjugacy, cocycle conjugacy, and outer conjugacy. Using the generalized determinant introduced by P. de la Harpe and G. Skandalis ([14]), we show that approximate conjugacy implies cocycle conjugacy when a (unital) C^*-algebra is simple and possesses a unique trace.

In Section 4 we consider product type $\mathbf{Z}^2$-actions on UHF algebras, i.e., pairs (α, β) of commuting automorphisms

$$\alpha \cong \bigotimes_{k=1}^{\infty} Adu_k, \ \ \beta \cong \bigotimes_{k=1}^{\infty} Adv_k.$$

Considering the case when the $n_k \times n_k$ matrices u_k, v_k commute at first, we show that the Rohlin property in this case is characterized by the property of uniform distribution of the joint spectral set $\mathrm{Sp}(\otimes_{k=m}^{n} u_k, \otimes_{k=m}^{n} v_k)$ $(m \leq n)$. From this we show that any such pairs are approximately conjugate.

We then investigate two special classes of UHF algebras. The first one is of the form $\otimes_{k=1}^{\infty} M_{p_k^{i_k}}$, where p_k $(k \in \mathbf{N})$ are primes and non-negative (finite) integers i_k $(k \in \mathbf{N})$ satisfy $\sum_{k=1}^{\infty} i_k = \infty$. The second one is of the form $\otimes_{k \in K} M_{q_k^\infty}$ with primes q_k $(k \in K)$ (where $\#K \leq \infty$) and $M_{q_k^\infty}$ of course means the infinite tensor product of M_{q_k}. For algebras in the first class we construct infinitely many non-cocycle conjugate product type $\mathbf{Z}^2$-actions with the Rohlin property. We would like to emphasize that for $\mathbf{Z}$-actions this phenomenon does not occur. On the other hand, for algebras in the second class we show that all the product type $\mathbf{Z}^2$-actions with the Rohlin property are mutually approximately conjugate. Combining these results we get the classification of product type $\mathbf{Z}^2$-actions with the Rohlin property on UHF algebras up to outer conjugacy.

2 Rohlin type theorem

Let N be a positive integer . We first define the Rohlin property for $\mathbf{Z}^N$-actions on unital C^*-algebras. As mentioned above this is a simple generalization of that in the case of $N = 1$ [15]. Let $\xi_1, \dots, \xi_N$ be the canonical basis of $\mathbf{Z}^N$ i.e.,

$$\xi_i = (0, \dots, 0, 1, 0, \dots, 0),$$

where 1 is in the i-th component, and let $I = (1, \dots, 1)$ throughout this section. For $m = (m_1, \dots, m_N)$ and $n = (n_1, \dots, n_N) \in \mathbf{Z}^N$, $m \leq n$ means $m_i \leq n_i$ for each $i = 1, \dots, N$. We define

$$m\mathbf{Z}^N = \{(m_1 n_1, \dots, m_N n_N) | (n_1, \dots, n_N) \in \mathbf{Z}^N\}$$

for $m = (m_1, \ldots, m_N) \in \mathbf{Z}^N$ and let $\mathbf{Z}^N$ act on $\mathbf{Z}^N/m\mathbf{Z}^N$ by addition modulo $m\mathbf{Z}^N$.

Definition 1 Let α be a $\mathbf{Z}^N$-action on a unital C^*-algebra A i.e., α is a group homomorphism from $\mathbf{Z}^N$ into the automorphisms $Aut(A)$ of A. Then α is said to have the *Rohlin property* if for any $m \in \mathbf{N}^N$ there exist $R \in \mathbf{N}$ and $m^{(1)}, \ldots, m^{(R)} \in \mathbf{N}^N$ with $m^{(1)}, \ldots, m^{(R)} \geq m$ and which satisfy the following condition: For any $\varepsilon > 0$ and finite subset F of A, there exist projections

$$e_g^{(r)} \quad \left(r = 1, \ldots, R,\ g \in \mathbf{Z}^N/m^{(r)}\mathbf{Z}^N\right)$$

in A satisfying

$$\sum_{r=1}^{R} \sum_{g \in \mathbf{Z}^N/m^{(r)}\mathbf{Z}^N} e_g^{(r)} = 1 \ ,$$
$$\|[x,\ e_g^{(r)}]\| < \varepsilon \ , \tag{1}$$
$$\|\alpha_{\xi_i}(e_g^{(r)}) - e_{\xi_i+g}^{(r)}\| < \varepsilon$$

for any $x \in F$, $r = 1, \ldots, R$, $i = 1, \ldots, N$, and $g \in \mathbf{Z}^N/m^{(r)}\mathbf{Z}^N$.

Remark 2 When A is a UHF algebra, using Christensen's perturbation argument ([5, Theorem 5.3.]), we can restate the definition of the Rohlin property as follows. For any $n, m \in \mathbf{N}$ with $1 \leq n \leq N$ there exist $R \in \mathbf{N}$ and positive integers $m^{(1)}, \ldots, m^{(R)} \geq m$ which satisfy the following condition: For any $\varepsilon > 0$ and finite subset F of A there exist projections

$$e_0^{(r)}, \ldots, e_{m^{(r)}-1}^{(r)} \quad (r = 1, \ldots, R)$$

in A satisfying

$$\sum_{r=1}^{R} \sum_{j=0}^{m^{(r)}-1} e_j^{(r)} = 1 \ ,$$
$$\|[x,\ e_j^{(r)}]\| < \varepsilon$$

for each $r = 1, \ldots, r,\ j = 0, \ldots, m^{(r)} - 1$ and $x \in F$, and

$$\|\alpha_{\xi_n}(e_j^{(r)}) - e_{j+1}^{(r)}\| < \varepsilon \ ,$$
$$\|\alpha_{\xi_{n'}}(e_j^{(r)}) - e_j^{(r)}\| < \varepsilon$$

for each $n' = 1, \ldots, N$ with $n' \neq n$, $r = 1, \ldots R$ and $j = 0, \ldots, m^{(r)} - 1$, where $e_{m^{(r)}}^{(r)} \equiv e_0^{(r)}$.

For automorphisms of C^*-algebras a notion of uniform outerness was introduced in [15]. That is, an automorphism α of a unital C^*-algebra A is said to be *uniformly outer* if for any $a \in A$, any nonzero projection $p \in A$ and any $\varepsilon > 0$, there exist projections $p_1, \dots, p_n$ in A such that

$$p = \sum_{i=1}^{n} p_i \ ,$$

$$\|p_i a \alpha(p_i)\| < \varepsilon \quad (i = 1, \dots, n) \ .$$

It was shown that this notion for automorphisms of UHF algebras is equivalent to the usual outerness for the automorphisms of the von Neumann algebras obtained through the GNS representations associated with the traces ([15, Theorem 4.5]). Based on this fact and the Rohlin type theorem for automorphisms of von Neumann algebras due to A. Connes ([4, Theorem 1.2.5]), a C^*-algebraic version of the theorem (for the UHF algebras) was shown by A. Kishimoto ([16, Theorem 1.3]). We extend Kishimoto's work to $\mathbf{Z}^2$-actions.

Theorem 3 *Let α be a $\mathbf{Z}^2$-action on a UHF algebra A. Then the following conditions are equivalent:*

(1) *α has the Rohlin property.*

(2) *α_g is uniformly outer for each $g \in \mathbf{Z}^2 \setminus \{0\}$.*

Once we establish this theorem, we have immediately

Corollary 4 *Let α be a $\mathbf{Z}^2$-action on a UHF algebra A. Then the following conditions are equivalent:*

(1) *α has the Rohlin property as a $\mathbf{Z}^2$-action on A.*

(2) *α_g has the Rohlin property as an automorphism of A for each $g \in \mathbf{Z}^2 \setminus \{0\}$.*

In Theorem 3 it is obvious that (1) implies (2). We devote the rest of this section to prove the converse in several steps.

Lemma 5 *Let α be a $\mathbf{Z}^2$-action on a UHF algebra A. If the condition (2) in Theorem 3 holds then for any $m = (m_1, m_2) \in \mathbf{N}^2$, $\varepsilon > 0$ and any unital full matrix subalgebra B of A, there exists an orthogonal family $(\ e_g \mid g \in \mathbf{Z}^2, 0 \le g \le m - I\ )$ of projections in $A \cap B'$ such that*

$$\|\alpha_{\xi_i}(e_g) - e_{\xi_i + g}\| < \varepsilon$$

for any $i = 1,\ 2$ and $g \in \mathbf{Z}^2$ with $\ 0 \le g, \xi_i + g \le m - I$, and furthermore,

$$1 < (|m| + 1)\tau(e_0) \ ,$$

where τ is the unique tracial state of A and $|m| \equiv m_1 \cdot m_2$.

Proof. Let (π_τ, H_τ) be the GNS representation associated with τ. By the uniqueness of a trace we can extend each α_g $(g \in \mathbf{Z}^2)$ to an automorphism of the AFD II_1 factor $\pi_\tau(A)''$ $(\subseteq B(H_\tau))$ and we use the same symbol α_g for this extension. Since α_g is outer on $\pi_\tau(A)''$ for $g \in \mathbf{Z}^2 \setminus \{0\}$ by [15, Theorem 4.5], it follows from [20, Theorem 2] that for any $m \in \mathbf{N}^2$ there exists a strongly central sequence

$$\Big((\, E_g^{(j)} \,|\, g \in \mathbf{Z}^2, 0 \le g \le m - I \,) \mid j \in \mathbf{N} \Big)$$

of orthogonal families of projections in $\pi_\tau(A)''$ such that

$$\sum_{\substack{g \in \mathbf{Z}^2 \\ 0 \le g \le m-I}} E_g^{(j)} = 1$$

for each $j \in \mathbf{N}$ and

$$\alpha_{\xi_i}(E_g^{(j)}) - E_{\xi_i + g}^{(j)} \longrightarrow 0$$

strongly as $j \to \infty$ for each $i = 1,\ 2$ and $g \in \mathbf{Z}^2$ with $0 \le g \le m - I$, where $(m_1, k) \equiv (0, k)$, $(k, m_2) \equiv (k, 0)$.

From this central sequence we shall construct a uniformly central sequence

$$\Big((\, f_g^{(j)} \,|\, g \in \mathbf{Z}^2,\ 0 \le g \le m - I \,) \mid j \in \mathbf{N} \Big)$$

of orthogonal families of projections in A such that

$$\pi_\tau \left(\sum_{\substack{g \in \mathbf{Z}^2 \\ 0 \le g \le m-I}} f_g^{(j)} \right) \longrightarrow 1$$

strongly as $j \to \infty$ and

$$\| \alpha_{\xi_i}(f_g^{(j)}) - f_{\xi_i + g}^{(j)} \| \longrightarrow 0$$

as $j \to \infty$ for each $i = 1,\ 2$ and $g \in \mathbf{Z}^2$ with $0 \le g, g + \xi_i \le m - I$. To do this, let $(\, A_j \,|\, j \in \mathbf{N} \,)$ be an increasing sequence of unital full matrix subalgebras of A such that $\cup_j A_j$ is dense in A. From [15, Lemma 4.7] we find a uniformly central sequence $(\, e_j \,|\, j \in \mathbf{N} \,)$ of projections in A such that

$$\pi_\tau(e_j) - E_0^{(j)} \longrightarrow 0$$

strongly as $j \to \infty$. Changing e_j slightly and taking a subsequence, we may assume that $e_j \in (\cup_k A_k) \cap A_j'$ for each j. Let $\varepsilon > 0$. From [5, Corollary 6.8], by taking inner perturbations, there are $\alpha_1, \alpha_2 \in Aut(A)$ such that

$$\| \alpha_i - \alpha_{\xi_i} \| < \varepsilon \ ,$$

$$\alpha_i\left(\bigcup_{j\in\mathbf{N}} A_j\right) \subseteq \bigcup_{j\in\mathbf{N}} A_j$$

for $i=1,2$. Set

$$h_j = e_j\left(\sum_{\substack{g=(g_1,g_2)\in\mathbf{Z}^2\setminus\{0\}\\ -(m-I)\le g\le m-I}} (\alpha_1)^{g_1}(\alpha_2)^{g_2}(e_j)\right)e_j\ ,$$

$$k_j = e_j\left(\sum_{\substack{g\in\mathbf{Z}^2\setminus\{0\}\\ -(m-I)\le g\le m-I}} \alpha_g(e_j)\right)e_j\ ,$$

$\eta_j=\tau(h_j)$ and $\kappa_j=\tau(k_j)$. Then it follows that

$$\|h_j-k_j\| < \varepsilon(m_1+m_2-2)\{(2m_1-1)(2m_2-1)-1\}\ .$$

Furthermore, $\lim_j \kappa_j = 0$ since

$$\begin{aligned}
\lim_{j\to\infty}\kappa_j &= \lim_{j\to\infty}\tau\left(e_j\sum_{\substack{g\in\mathbf{Z}^2\setminus\{0\}\\ -(m-I)\le g\le m-I}}\alpha_g(e_j)\right)\\
&\le \lim_{j\to\infty}\sum_{\substack{g,\ h\in\mathbf{Z}^2,\ g\neq h\\ 0\le g,\ h\le m-I}}\tau(e_j\alpha_{g-h}(e_j))\\
&\le \lim_{j\to\infty}\sum_{\substack{g,\ h\in\mathbf{Z}^2,\ g\neq h\\ 0\le g,\ h\le m-I}}\tau(\alpha_h(e_j)\alpha_g(e_j))\\
&= \lim_{j\to\infty}\sum_{\substack{g,\ h\in\mathbf{Z}^2,\ g\neq h\\ 0\le g,\ h\le m-I}}\tau(E_h^{(j)}E_g^{(j)}) = 0\ .
\end{aligned}$$

Let p_j be the spectral projection of h_j corresponding to $(0,\ \eta_j^{\frac{1}{2}})$. Then $p_j\in A$ since $\mathrm{Sp}(h_j)$ is finite. $p_j\le e_j$ and ${\eta_j}^{\frac{1}{2}}(e_j-p_j)\le h_j$ because ${\eta_j}^{\frac{1}{2}}\chi_{[{\eta_j}^{\frac{1}{2}},\ \infty)}(t)\le t$ $(t\in[0,\ \infty))$, hence

$$\tau(e_j)-{\eta_j}^{\frac{1}{2}}\le\tau(p_j)\le\tau(e_j)\ .$$

In addition

$$\|p_j \left(\sum_{\substack{g=(g_1,g_2)\in \mathbf{Z}^2\setminus\{0\} \\ -(m-I)\le g\le m-I}} (\alpha_1)^{g_1}(\alpha_2)^{g_2}(p_j) \right) p_j\| \le \|p_j h_j p_j\| \le \eta_j{}^{\frac{1}{2}}.$$

So for any $g, h \in \mathbf{Z}^2$ with $0 \le g, h \le m - I$ and $g \ne h$, we have

$$\begin{aligned}
\|\alpha_g(p_j)\alpha_h(p_j)\|^2 &= \|\alpha_g(p_j\alpha_{h-g}(p_j))\|^2 = \|p_j\alpha_{h-g}(p_j)\|^2 \\
&= \|p_j\alpha_{h-g}(p_j)p_j\| \\
&\le \|p_j \left(\sum_{\substack{g\in \mathbf{Z}^2\setminus\{0\} \\ -(m-I)\le g\le m-I}} \alpha_g(p_j) \right) p_j\| \\
&\le \varepsilon' + \eta_j{}^{\frac{1}{2}},
\end{aligned}$$

where $\varepsilon' = \varepsilon(m_1 + m_2 - 2)\{(2m_1 - 1)(2m_2 - 1) - 1\}$. Here $\lim \eta_j = 0$ since $\lim \kappa_j = 0$ and $\lim \|h_j - k_j\| = 0$. Therefore taking a sufficiently large j for each $\varepsilon > 0$, we obtain the required $f_g^{(j)}$ near $\alpha_g(p_j)$ by slight modification.

Noting that $\sum_{0\le g\le m-I} \tau(f_g^{(j)}) \to 1$ and $\tau(f_g^{(j)}) = \tau(f_h^{(j)})$, we have

$$\tau(f_g^{(j)}) \longrightarrow \frac{1}{|m|} \ .$$

Furthermore for any unital full matrix subalgebra B of A, taking a sufficiently large j, we may assume that $f_g^{(j)} \in A \cap B'$ for any g. This concludes the proof. ∎

In Ocneanu's result [20, Theorem 2], applied in the above proof, we have the cyclicity condition (under the action) of the projections $(\, E_g^{(j)} \,|\, g \in \mathbf{Z}^2, 0 \le g \le m - I \,)$ in the von Neumann algebra $\pi_\tau(A)''$. However when approximating these projections by the projections $(\, f_g^{(j)} \,|\, g \in \mathbf{Z}^2, 0 \le g \le m - I \,)$ in the C^*-algebra A, we lose the cyclicity condition. It is our next problem to restore this cyclicity condition. To do this we need a technical lemma from [16]. Let $K(l^2(\mathbf{Z}))$ be the compact operators on $l^2(\mathbf{Z})$ and let $(\, E_{i,j} \,|\, i, j \in \mathbf{Z} \,)$ be the canonical matrix units for $K(l^2(\mathbf{Z}))$. On $K(l^2(\mathbf{Z}))$ we define an automorphism σ by $\sigma(E_{i,j}) = E_{i+1,j+1}$ $(i,\ j \in \mathbf{Z})$. For any $n, k, l \in \mathbf{N}$ with $1 < k < l$, define

$$N = n(2k + l - 1) \ ,$$

$$f = \sum_{i=1}^{k-1} \left(\frac{i}{k} E_{ni,ni} + \frac{k-i}{k} E_{n(k+l+i),n(k+l+i)} + \frac{\sqrt{i(k-i)}}{k} E_{ni,n(k+l+i)} \right.$$

$$+ \frac{\sqrt{i(k-i)}}{k} E_{n(k+l+i),ni} \Bigg) + \sum_{i=k}^{k+l} E_{ni,ni} \ , \tag{2}$$

$$e_i = \sigma^{i-n}(f) \ \ (i = 0, \ldots, n-1) \ .$$

Then $(e_i \,|\, i = 0, \ldots, n-1)$ is an orthogonal family of projections in $K(l^2(\mathbf{Z}))$. Hence for any $\varepsilon > 0$, there exist k, l with $1 \ll k \ll l$ such that

$$\sum_{i=0}^{n-1} e_i \le P_N \ \ (P_N \equiv \sum_{i=0}^{N-1} E_{i,i}) \ ,$$

$$\|\sigma(e_i) - e_{i+1}\| < \varepsilon \ \ (i = 0, \ldots, n-1, \ \ e_n \equiv e_0) \ ,$$

$$\frac{n \ \dim e_0}{N} > 1 - \varepsilon$$

(see [16, Lemma 2.1] for the detail). Using these estimates we have the next lemma.

Lemma 6 *Let α be a $\mathbf{Z}^2$-action on a UHF algebra A. If α_g is uniformly outer for any $g \in \mathbf{Z}^2 \setminus \{0\}$, then for any $m \in \mathbf{N}$, $\varepsilon > 0$ and any unital full matrix subalgebra B of A there exists an orthogonal family $(e_i \,|\, i = 0, \ldots, m-1)$ of projections in $A \cap B'$ such that*

$$\|\alpha_{\xi_1}(e_i) - e_{i+1}\| < \varepsilon \ ,$$

$$\|\alpha_{\xi_2}(e_i) - e_i\| < \varepsilon \ ,$$

$$\tau(1 - \sum_{i=0}^{m-1} e_i) \le \varepsilon \tau(e_0)$$

for $i = 0, \ldots, m-1$, where $e_m \equiv e_0$.

Proof. Let $m \in \mathbf{N}$, $\varepsilon_1 > 0$ and let B_1 be a unital full matrix subalgebra of A. By the above statement there exist $k_1, \ l_1 \in \mathbf{N}$ with $1 \ll k_1 \ll l_1$ and an orthogonal family $(e_i \,|\, i = 0, \ldots, m-1)$ of projections in $K(l^2(\mathbf{Z}))$ such that

$$\sum_{i=0}^{m-1} e_i \le P_{N_1} \ ,$$

$$\|\sigma(e_i) - e_{i+1}\| < \varepsilon_1 \ \ (i = 0, \ldots, m-1) \ ,$$

$$\frac{m \ \dim e_0}{N_1} > 1 - \varepsilon_1 \ , \tag{3}$$

where $N_1 \equiv m(2k_1 + l_1 - 1)$ and $e_m \equiv e_0$. Similarly by the above statement (for $n = 1, \ \varepsilon_1$ and B_1), there exist $k_2, \ l_2 \in \mathbf{N}$ with $1 \ll k_2 \ll l_2$ and a projection e in $K(l^2(\mathbf{Z}))$ such that

$$e \le P_{N_2} \ ,$$

$$\|\sigma(e) - e\| < \varepsilon_1 \ ,$$
$$\frac{\dim e}{N_2} > 1 - \varepsilon_1 \ , \tag{4}$$

where $N_2 \equiv 2k_2 + l_2 - 1$.

Next by applying Lemma 5 to $(N_1, N_2) \in \mathbf{N}^2$, any $\varepsilon_2 > 0$ and B_1, there exists an orthogonal family $(\, p_g \,|\, g \in \mathbf{Z}^2, 0 \le g \le (N_1 - 1, N_2 - 1)\,)$ of projections in $A \cap B_1'$ such that

$$\|\alpha_{\xi_i}(p_g) - p_{\xi_i + g}\| < \varepsilon_2 \ ,$$
$$1 \le (N_1 N_2 + 1)\tau(p_0) \tag{5}$$

for any $i = 1,\ 2$ and $g \in \mathbf{Z}^2$ with $0 \le g, \xi_i + g \le (N_1 - 1, N_2 - 1)$.

If we put

$$x_1 = \frac{1}{N_2} \sum_{j=0}^{N_2 - 1} \left\{ \sum_{i=0}^{N_1 - 2} p_{(i+1,j)} \alpha_{\xi_1}(p_{(i,j)}) \right.$$
$$\left. + (1 - \sum_{i=1}^{N_1 - 1} p_{(i,j)})(1 - \sum_{i=0}^{N_1 - 2} \alpha_{\xi_1}(p_{(i,j)})) \right\} \ ,$$

then we have

$$x_1 \alpha_{\xi_1}(p_{(i,j)}) = p_{(i+1,j)} x_1$$

for any $i = 0, \ldots, N_1 - 2,\ j = 0, \ldots, N_2 - 1$ and

$$x_1 - 1 = \frac{1}{N_2} \sum_{j=0}^{N_2 - 1} \left\{ \sum_{i=0}^{N_1 - 2} p_{(i+1,j)} \left(\alpha_{\xi_1}(p_{(i,j)}) - p_{(i+1,j)} \right) \right.$$
$$\left. + (1 - \sum_{i=1}^{N_1 - 1} p_{(i,j)})(- \sum_{i=0}^{N_1 - 2} \alpha_{\xi_1}(p_{(i,j)})) \right\} \ .$$

Noting that $\|\alpha_{\xi_1}(p_{(i,j)}) - p_{(i+1,j)}\| < \varepsilon_2$, we have $\|x_1 - 1\| < 2(N_1 - 1)\varepsilon_2$. So taking the polar decomposition $u_1|x_1|$ of x_1 for a sufficiently small $\varepsilon_2 > 0$, we obtain a unitary u_1 with $\|u_1 - 1\| < 4(N_1 - 1)\varepsilon_2$. By the uniqueness of the polar decomposition we have

$$Ad\, u_1 \circ \alpha_{\xi_1}(p_{(i,j)}) = p_{(i+1,j)}$$

for $i = 0, \ldots, N_1 - 2,\ j = 0, \ldots, N_2 - 1$. Similarly for α_{ξ_2}, we obtain a unitary u_2 in A such that

$$\|u_2 - 1\| < 4(N_2 - 1)\varepsilon_2 \ ,$$
$$Ad\, u_2 \circ \alpha_{\xi_2}(p_{(i,j)}) = p_{(i,j+1)}$$

for $i = 0, \ldots, N_1 - 1,\ j = 0, \ldots, N_2 - 2$. Let $\alpha_1 = Ad\, u_1 \circ \alpha_{\xi_1}$ and let $\alpha_2 = Ad\, u_2 \circ \alpha_{\xi_2}$. Since $[p_{(0,0)}] = [p_{(1,0)}]$ it follows that there exists a partial isometry

v_1 of $A\cap B_1'$ such that $v_1^*v_1 = p_{(0,0)}$ and $v_1v_1^* = p_{(1,0)}$. Similarily there exists a partial isometry v_2 of $A\cap B_1'$ such that $v_2^*v_2 = p_{(0,0)}$ and $v_2v_2^* = p_{(0,1)}$. Then $Ad\, v_2^*\circ\alpha_2(p_{(0,0)}) = p_{(0,0)}$, so $Ad\, v_2^*\circ\alpha_2 \in Aut(p_{(0,0)}Ap_{(0,0)})$. On the other hand $\alpha_{\xi_2}\in Aut(A)$ has the Rohlin property as a single automorphism, and hence so does $Ad\, v_2^*\circ\alpha_2$. Therefore $Ad\, v_2^*\circ\alpha_2$ is stable by [12, 7]. More precisely for any $\varepsilon_3 > 0$, any unital full matrix subalgebra B_2 of A and the unitary $v_2^*\alpha_2(v_1)^*\alpha_1(v_2)v_1 \in p_{(0,0)}Ap_{(0,0)}$, if B_1 is taken sufficiently large in advance, we have a unitary w in $A\cap B_2'$ such that

$$\|v_2^*\alpha_2(v_1)^*\alpha_1(v_2)v_1 - (Ad\, v_2^*\circ\alpha_2(w))\cdot w^*\| < \varepsilon_3.$$

Let $w_1 = v_1w$ and let $w_2 = v_2$. Then w_1 and w_2 are partial isometries in $A\cap B_2'$ such that $w_1^*w_1 = w_2^*w_2 = p_{(0,0)}$, $w_1w_1^* = p_{(1,0)}$, $w_2w_2^* = p_{(0,1)}$ and

$$\|\alpha_1(w_2)w_1 - \alpha_2(w_1)w_2\| = \|v_2^*\alpha_2(v_1)^*(\alpha_1(v_2)v_1w - \alpha_2(v_1w)v_2)w^*\| \le \varepsilon_3 \quad (6)$$

Define

$$E_{i,j}^{(k)} = \begin{cases} \alpha_2^{i-1}(\alpha_1^k(w_2))\alpha_2^{i-2}(\alpha_1^k(w_2))\cdots\alpha_2^{j}(\alpha_1^k(w_2)) & (\,i>j\,) \\ p_{(k,i)} & (\,i=j\,) \\ \alpha_2^{i}(\alpha_1^k(w_2))^*\alpha_2^{i+1}(\alpha_1^k(w_2))^*\cdots\alpha_2^{j-1}(\alpha_1^k(w_2))^* & (\,i<j\,) \end{cases}$$

for $k = 0,1,\ i,j = 0,\ldots,N_2-1$. Then we can easily see that $(\,E_{i,j}^{(k)}\mid i,j = 0,\ldots,N_2-1\,)$ is a system of matrix units. For any unital full matrix subalgebra B_3 of A, by taking a sufficiently large B_2 including B_3, we may assume that $\{\,E_{i,j}^{(k)}\mid k=0,1;\ i,j=0,\ldots,N_2-1\,\}\subseteq A\cap B_3'$. Let $C^{(k)}$ be the C^*-subalgebra of A generated by $\{\,E_{i,j}^{(k)}\mid i,j = 0,\ldots,N_2-1\,\}$ and let Φ_k be the canonical isomorphism from $C^{(k)}$ onto $P_{N_2}K(l^2(\mathbf{Z}))P_{N_2}$. Define

$$e^{(k)} = \Phi_k^{-1}(e)\ .$$

Since $\sigma\circ\Phi_k = \Phi_k\circ\alpha_2\lceil C^{(k)}$, we have from (4) that

$$e^{(k)} \le \sum_{i=0}^{N_2-1} p_{(k,i)}\ ,$$

$$\|\alpha_2(e^{(k)}) - e^{(k)}\| < \varepsilon_1\ ,$$
$$\tau(e^{(k)}) > (1-\varepsilon_1)N_2\tau(p_{(0,0)})\ . \quad (7)$$

Furthermore define

$$W_1 = \left(\sum_{i=0}^{N_2-1}\alpha_2^i(w_1)\right)e^{(0)}\ ,$$

$$e^{(1)\prime} = W_1W_1^* \le \sum_{i=0}^{N_2-1} p_{(1,i)}\ .$$

Again for any unital full matrix subalgebra B_4 of A, by taking a sufficiently large B_3 including B_4, we may assume that $W_1 \in A \cap B_4'$. Then recalling the formula (2), we have

$$e^{(1)\prime} = \left(\sum_{i=0}^{N_2-1} \alpha_2^i(w_1) \right) X \left(\sum_{i'=0}^{N_2-1} \alpha_2^{i'}(w_1) \right)^* , \tag{8}$$

where X denotes

$$\sum_{j=1}^{k_2-1} \left\{ \frac{j}{k_2} E_{j,j}^{(0)} + \frac{k_2-j}{k_2} E_{k_2+l_2+j,k_2+l_2+j}^{(0)} + \frac{\sqrt{j(k_2-j)}}{k_2} E_{j,k_2+l_2+j}^{(0)} \right.$$

$$\left. + \frac{\sqrt{j(k_2-j)}}{k_2} E_{k_2+l_2+j,j}^{(0)} \right\} + \sum_{j=k_2}^{k_2+l_2} E_{j,j}^{(0)} .$$

On the right hand side of (8), the nonzero terms are calculated as follows:

$$\begin{aligned} \alpha_2^j(w_1) E_{j,j}^{(0)} \alpha_2^j(w_1)^* &= \alpha_2^j(w_1) p_{(0,j)} \alpha_2^j(w_1)^* \\ &= p_{(1,j)} \\ &= E_{j,j}^{(1)} , \end{aligned}$$

$$\begin{aligned} &\alpha_2^j(w_1) E_{j,k_2+l_2+j}^{(0)} \alpha_2^{k_2+l_2+j}(w_1)^* \\ &\quad = \alpha_2^j(w_1)\alpha_2^j(w_2)^* \cdots \alpha_2^{k_2+l_2+j-1}(w_2^*\alpha_2(w_1)^*) \\ &\quad \overset{\varepsilon_3}{\approx} \alpha_2^j(w_1)\alpha_2^j(w_2)^* \cdots \alpha_2^{k_2+l_2+j-1}(w_1^*\alpha_1(w_2)^*) , \end{aligned}$$

where $x \overset{\varepsilon}{\approx} y$ means $\|x-y\| < \varepsilon$. Applying (6) repeatedly we have

$$\begin{aligned} &\alpha_2^j(w_1) E_{j,k_2+l_2+j}^{(0)} \alpha_2^{k_2+l_2+j}(w_1)^* \\ &\overset{(k_2+l_2)\varepsilon_3}{\approx} E_{j,k_2+l_2+j}^{(1)} . \end{aligned}$$

We estimate the other nonzero terms similarly and obtain

$$\|e^{(1)\prime} - e^{(1)}\| \le \sum_{j=1}^{k_2-1} \frac{\sqrt{j(k_2-j)}}{k_2} (k_2+l_2)\varepsilon_3 .$$

Let $c_1(k_2, l_2, \varepsilon_3)$ be the right hand side of the above inequality. Then we have

$$\begin{aligned} \|\alpha_2(e^{(1)\prime}) - e^{(1)}\| &\le 2\|e^{(1)\prime} - e^{(1)}\| + \|\alpha_2(e^{(1)}) - e^{(1)}\| \\ &\le 2c_1(k_2, l_2, \varepsilon_3) + \varepsilon_1 . \end{aligned}$$

For any $\varepsilon_4 > 0$ and any unital full matrix subalgebra B_5 of A, applying [15, Lemma 3.5] and taking a sufficiently large B_4 including B_5, we have a partial isometry W_1' of $A \cap B_5'$ such that $(W_1')^* W_1' = e^{(0)}$, $W_1'(W_1')^* = e^{(1)\prime}$ and

$$\begin{aligned} \|\alpha_2(W_1') - W_1'\| &\leq \|\alpha_2(e^{(0)}) - e^{(0)}\| + \|\alpha_2(e^{(1)\prime}) - e^{(1)\prime}\| + \varepsilon_4 \\ &\leq \varepsilon_1 + 2c_1(k_2, l_2, \varepsilon_3) + \varepsilon_1 + \varepsilon_4 \ . \end{aligned} \tag{9}$$

Of course we can make the last quantity very small. By using this W_1', let D be the C^*-subalgebra of A generated by

$$\{\alpha_1^{i-1}(W_1)\alpha_1^{i-2}(W_1') \cdots \alpha_1^{j}(W_1') \mid N_1 - 1 \geq i > j \geq 0 \} \ .$$

Here again for any unital full matrix subalgebra B_6, by taking a sufficiently large B_5 including B_6, we may assume that $D \subseteq A \cap B_6'$. As D is isomorphic to $P_{N_1} K(l^2(\mathbf{Z}))P_{N_1}$, let Ψ be the canonical isomorphism from D onto $P_{N_1} K(l^2(\mathbf{Z}))P_{N_1}$ and let

$$f_i \equiv \Psi^{-1}(e_i)$$

for $i = 0, \ldots, m-1$. Then $(\ f_i \mid i = 0, \ldots, m-1\ )$ is an orthogonal family of projections in $A \cap B_6'$ such that

$$\begin{aligned} \|\alpha_1(f_i) - f_{i+1}\| &< \varepsilon_1 \ , \\ m\tau(f_0) &> (1 - \varepsilon_1)N_1\tau(e^{(0)}) \end{aligned} \tag{10}$$

for $i = 0, \ldots, m-1$, where $f_m \equiv f_0$. Thus we have for $i = 0, \ldots, m-1$,

$$\begin{aligned} \|\alpha_{\xi_1}(f_i) - f_{i+1}\| &\leq \|\alpha_{\xi_1}(f_i) - \alpha_1(f_i)\| + \|\alpha_1(f_i) - f_{i+1}\| \\ &\leq 2\|u_1 - 1\| + \varepsilon_1 \\ &\leq 2 \cdot 4(N_1 - 1)\varepsilon_2 + \varepsilon_1 \ . \end{aligned}$$

Using the formula (2), the formula (9) and

$$\begin{aligned} \|\alpha_1\alpha_2 - \alpha_2\alpha_1\| &\leq 2(\|\alpha_1 - \alpha_{\xi_1}\| + \|\alpha_2 - \alpha_{\xi_2}\|) \\ &\leq 4(\|u_1 - 1\| + \|u_2 - 1\|) \ , \end{aligned}$$

we can also make $\|\alpha_{\xi_2}(f_i) - f_i\|$ very small. Finally we want to estimate $\tau(f_0)$. We have already three inequalities from (5),(7) and (10)

$$1 \leq (N_1N_2 + 1)\tau(p_{(0,0)}) \ ,$$

$$\tau(e^{(0)}) > (1 - \varepsilon_1)N_2\tau(p_{(0,0)}) \ ,$$

$$m\tau(f_0) > (1 - \varepsilon_1)N_1\tau(e^{(0)}) \ .$$

From these we obtain

$$m\tau(f_0) > (1 - \varepsilon_1)^2 N_1N_2 \frac{1}{N_1N_2 + 1} \left(m\tau(f_0) + \tau(1 - \sum_{i=0}^{m-1} f_i) \right) \ .$$

Since $1 \ll k_i \ll l_i$, $(\, f_i \,|\, i = 0, \dots, m-1\,)$ satisfies the desired conditions. ∎

Proof of Theorem 3.

Let α be a $\mathbf{Z}^2$-action on a UHF algebra A which satisfies the condition (2). For any $m \in \mathbf{N}$ we take $m_0,\ m_1 \in \mathbf{N}$ such that $m \ll m_1 \ll m_0$ and m_0 is divided by m_1. Furthermore for any $\varepsilon_1 > 0$ and finite subset F of A, we take a unital full matrix subalgebra B_1 of A such that for any $x \in F$ there exists $y \in B_1$ with $\|x - y\| < \varepsilon_1$. If we apply Lemma 6 to any $n \in \mathbf{N}$ and $\varepsilon_2 > 0$ then we have an orthogonal family $(\ e_i \mid i = 0, \dots, m-1\ )$ of projections in $A \cap B_1'$ satisfying

$$\|\alpha_{\xi_1}(e_i) - e_{i+1}\| < \varepsilon_2 \ ,$$

$$\|\alpha_{\xi_2}(e_i) - e_i\| < \varepsilon_2 \ ,$$

$$\tau(e_0) \geq n\tau(1 - \sum_{i=0}^{m_0-1} e_i)$$

for $i = 0, \dots, m_0 - 1$, where $e_{m_0} \equiv e_0$. This is not sufficient because the sum $\sum_{i=0}^{m-1} e_i$ of the projections (e_i) may not be 1. We will cope with this problem now. Put

$$x_1 = \sum_{i=0}^{m_0-1} e_{i+1}\alpha_{\xi_1}(e_i) + \left(1 - \sum_{i=0}^{m_0-1} e_i\right)\left(1 - \sum_{i=0}^{m_0-1} \alpha_{\xi_1}(e_i)\right) \ ,$$

$$x_2 = \sum_{i=0}^{m_0-1} e_i\alpha_{\xi_2}(e_i) + \left(1 - \sum_{i=0}^{m_0-1} e_i\right)\left(1 - \sum_{i=0}^{m_0-1} \alpha_{\xi_2}(e_i)\right)$$

and let $u_1|x_1|$ and $u_2|x_2|$ be the polar decompositions of x_1 and x_2 respectively. As in the proof of Lemma 6 we can show that u_1 and u_2 are unitaries in A satisfying

$$\|u_1 - 1\| < 4m_0\varepsilon_2 \ ,$$

$$Ad\,u_1 \circ \alpha_{\xi_1}(e_i) = e_{i+1}$$

for $i = 0, \dots, m_0 - 1$, where $e_{m_0} \equiv e_0$, and

$$\|u_2 - 1\| < 4m_0\varepsilon_2 \ ,$$

$$Ad\,u_2 \circ \alpha_{\xi_2}(e_i) = e_i$$

for $i = 0, \dots, m_0 - 1$. Let $\alpha_1 = Ad\,u_1 \circ \alpha_{\xi_1}$ and let $\alpha_2 = Ad\,u_2 \circ \alpha_{\xi_2}$. Then $\alpha_1^{m_0}$ and α_2 are automorphisms of e_0Ae_0. By Lemma 6 there are an orthogonal family $(\ p_j \mid j = 0, \dots, n-1\ )$ of projections in $A \cap B_1'$ and a positive number $c_1(m_0, \varepsilon_2)$ which decreases to zero as $\varepsilon_2 \to 0$ such that

$$\sum_{i=0}^{n-1} p_i \leq e_0 \ ,$$

$$\|\alpha_1^{m_0}(p_i) - p_{i+1}\| < c_1(m_0, \varepsilon_2) \ ,$$
$$\|\alpha_2(p_i) - p_i\| < c_1(m_0, \varepsilon_2)$$

for $i = 0, \ldots, n-1$, where $p_n \equiv p_0$, and

$$\tau(p_i) = \tau(1-e) \ ,$$

where $e \equiv \sum_{i=0}^{m_0-1} e_i$. We have used the fact $\tau(e_0) \geq n\tau(1-e)$ here. For any $\varepsilon_3 > 0$ and any unital full matrix subalgebra B_2 of A, by taking a sufficiently large B_1 and by applying [15, Lemma 3.5], there exists a partial isometry $v \in A \cap B_2'$ such that

$$v^*v = 1-e, \quad vv^* = p_0 \ ,$$

$$\begin{aligned} \|\alpha_2(v) - v\| &\leq \|\alpha_2(p_0) - p_0\| + \|\alpha_2(1-e) - (1-e)\| + \varepsilon_3 \\ &\leq c_1(m_0, \varepsilon_2) + m_0\varepsilon_2 + \varepsilon_3 \ . \end{aligned}$$

As before there also exists a unitary $u_1' \in A$ satisfying that $\|u_1' - 1\| < 4c_1(m_0, \varepsilon_2)$ and

$$Ad\, u_1' \circ \alpha_1^{m_0}(p_i) = p_{i+1}$$

for $i = 0, \ldots, n-1$. Let $\beta = Ad\, u_1' \circ \alpha_1^{m_0}$ and let $w = n^{-\frac{1}{2}} \sum_{i=0}^{n-1} \beta^i(v)$. Then we have

$$\begin{aligned} w^*w = 1-e, \quad ww^* \leq e_0 \ , \\ \|\beta(w) - w\| < n^{-\frac{1}{2}} \cdot 2 \ , \\ \|\alpha_2(w) - w\| < c_2(m_0, n, \varepsilon_2, \varepsilon_3) \end{aligned} \tag{11}$$

for some positive number $c_2(m_0, n, \varepsilon_2, \varepsilon_3)$ which we can make very small. Furthermore by taking B_2 very large we may assume that $w \in A \cap B_3'$ for any unital full matrix subalgebra B_3 of A. Let

$$E_{i,j} = \begin{cases} \alpha_1^{i-1}(w)\alpha_1^{i-2}(w) \cdots \alpha_1^{j}(w) & (if \ i > j) \\ \alpha_1^{i-1}(ww^*) & (if \ i = j) \\ \alpha_1^{i}(w)^*\alpha_1^{i+1}(w)^* \cdots \alpha_2^{j-1}(w)^* & (if \ i < j) \end{cases}$$

for $0 \leq i, \ j \leq m_0$ and let C be the C^*-subalgebra of A generated by $\{ E_{i,j} \,|\, 0 \leq i, j \leq m_0 - 1 \}$. Then C is isomorphic to M_{m_0+1} and we may assume that C is a subalgebra of $A \cap B_4'$ for any unital full matrix subalgebra B_4 of A if B_3 is very large. Let

$$U = \begin{bmatrix} 1 & & & & & & \\ & 0 & \cdot & \cdots & \cdot & 0 & 1 \\ & 1 & 0 & \cdots & \cdot & & 0 \\ & 0 & 1 & \cdot & & & \cdot \\ & \vdots & & \ddots & \ddots & & \vdots \\ & \cdot & & 0 & 1 & 0 & 0 \\ & 0 & & & 0 & 1 & 0 \end{bmatrix} \in M_{m_0+1} \ .$$

By simple calculation $\alpha_1 \lceil C = Ad\, U$ and

$$\mathrm{Sp}(U) = \{1\} \cup \{\, e^{\frac{2\pi i k}{m_0}} \,|\, k = 0, \ldots, m_0 - 1 \,\} \ .$$

Define orthogonal families $(\, e_i^{(1)} \,|\, i = 0, \ldots, m_1 - 1\,)$, $(\, e_j^{(2)} \,|\, j = 0, \ldots, m_1\,)$ of projections in $A \cap B_4'$ as follows:

$$\begin{aligned} e_i^{(1)} &\equiv \sum_{k=0}^{\frac{m_0}{m_1}-2} E_{m_1 + i(\frac{m_0}{m_1}-1)+k,\ m_1 + i(\frac{m_0}{m_1}-1)+k} \ , \\ e_j^{(2)} &\equiv E_{j,j} \ . \end{aligned}$$

Since $(\, E_{i,j} \,|\, 0 \le i, j \le m_0 - 1\,)$ is a system of matrix units in $A \cap B_4'$, there are canonical systems $(\, e_{k,l}^{(1)} \,|\, 0 \le k, l \le m_1 - 1\,)$, $(\, e_{k,l}^{(2)} \,|\, 0 \le k, l \le m_1\,)$ of matrix units associated with $(\, e_k^{(1)}\,)$ and $(\, e_k^{(2)}\,)$ respectively. Define a partial isometry V in $A \cap B_4'$ by

$$V = \sum_{i=0}^{m_1 - 1} e_{i+1,i}^{(1)} + \sum_{j=0}^{m_1} e_{j+1,j}^{(2)} \ ,$$

where $e_{m_1, m_1 - 1}^{(1)} \equiv e_{0, m_1 - 1}^{(1)}$ and $e_{m_1+1, m_1}^{(2)} \equiv e_{0, m_1}^{(2)}$. Then

$$\sum_{i=0}^{m_1-1} e_i^{(1)} + \sum_{j=0}^{m_1} e_j^{(2)} = 1_C \quad \left(= \sum_{i=0}^{m_0} E_{i,i} \right) \ ,$$

$$Ad\, V(e_i^{(1)}) = e_{i+1}^{(1)} \ ,$$

$$Ad\, V(e_j^{(2)}) = e_{j+1}^{(2)}$$

for $i = 0, \ldots, m_1 - 1$ and $j = 0, \ldots, m_1$ where $e_{m_1}^{(1)} \equiv e_0^{(1)}$ and $e_{m_1+1}^{(2)} \equiv e_0^{(2)}$. By a simple calculation

$$\mathrm{Sp}(V) = \{\, e^{\frac{2\pi i k}{m_0 - m_1}} \,|\, k = 0, \ldots, m_0 - m_1 - 1 \,\} \cup \{\, e^{\frac{2\pi i l}{m_1 + 1}} \,|\, l = 0, \ldots, m_1 \,\} \ .$$

Therefore $\mathrm{Sp}(V)$ is very close to $\mathrm{Sp}(U)$ if m_0 and m_1 are very large, i.e., V is almost unitarily equivalent to U. Consequently we can find orthogonal families $(\, f_i^{(1)} \,|\, i = 0, \ldots, m_1 - 1\,)$, $(\, f_j^{(2)} \,|\, j = 0, \ldots, m_1\,)$ of projections in $A \cap B_4'$ and a small enough positive number $c_3(m_0, m_1, \varepsilon_2, \varepsilon_3)$ in such a way that

$$\sum_{i=0}^{m_1-1} f_i^{(1)} + \sum_{j=0}^{m_1} f_j^{(2)} = 1_C \ ,$$

$$\|\alpha_1(f_i^{(1)}) - f_{i+1}^{(1)}\| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3) \ ,$$

$$\|\alpha_2(f_i^{(1)}) - f_i^{(1)}\| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3) \ ,$$
$$\|\alpha_1(f_j^{(2)}) - f_{j+1}^{(2)}\| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3) \ ,$$
$$\|\alpha_2(f_j^{(2)}) - f_j^{(2)}\| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3)$$

for $i = 0, \ldots, m_1 - 1$ and $j = 0, \ldots, m_1$, where $f_{m_1}^{(1)} \equiv f_0^{(1)}$ and $f_{m_1+1}^{(2)} \equiv f_0^{(2)}$. Then by considering (11), if n is very large,

$$(\, f_i^{(1)} \,|\, i = 0, \ldots, m_1 - 1 \,),$$
$$(\, f_j^{(2)} \,|\, j = 0, \ldots, m_1 \,),$$
$$(\, e_i - \alpha_1^i(ww^*) \,|\, i = 0, \ldots, m_0 - 1 \,)$$

satisfy the conditions appearing in Remark 2 (for $n = 1$). Hence α has the Rohlin property. ∎

3 Conjugacy

In this section we introduce three notions of conjugacy for $\mathbf{Z}^N$-actions on C^*-algebras and discuss their relationship. First we prepare some notations. For $\mathbf{Z}^N$-actions α, β on a unital C^*-algebra A, we write $\alpha \overset{\gamma,\varepsilon}{\approx} \beta$ when

$$\|\alpha_{\xi_i} - \gamma \circ \beta_{\xi_i} \circ \gamma^{-1}\| \leq \varepsilon \quad (i = 1, \ldots, N)$$

for $\varepsilon \geq 0$ and $\gamma \in Aut(A)$. For simplicity $\overset{\gamma,0}{\approx}$ will be denoted by $\overset{\gamma}{\cong}$ or $\cong$. Recall that a 1-cocycle for α means a mapping u from $\mathbf{Z}^N$ into the unitaries $U(A)$ of A satisfying $u_{g+h} = u_g \alpha_g(u_h)$ for each $g, h \in \mathbf{Z}^N$.

Definition 7 Let α and β be $\mathbf{Z}^N$-actions on a unital C^*-algebra A.
(1) α and β are *approximately conjugate* if for any $\varepsilon > 0$ there exists an automorphism γ of A such that $\alpha \overset{\gamma,\varepsilon}{\approx} \beta$.
(2) α and β are *cocycle conjugate* if there exist an automorphism γ of A and a 1-cocycle u for α such that

$$Ad\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1}$$

for each $g \in \mathbf{Z}^N$.
(3) α and β are *outer conjugate* if there exist an automorphism γ of A and unitaries $u_1, \ldots, u_N$ in A such that

$$Ad\, u_i \circ \alpha_{\xi_i} = \gamma \circ \beta_{\xi_i} \circ \gamma^{-1}$$

for $i = 1, \ldots, N$.

Cocycle conjugacy of course implies outer conjugacy, and we have

Proposition 8 *Assume that A is a simple separable unital C^*-algebra with a unique tracial state. Then approximately conjugate $\mathbf{Z}^N$-actions on A are cocycle conjugate.*

Our proof is based on the generalization of the determinant introduced by P. de la Harpe and G. Skandalis (see [14] for details) and the famous 2×2 matrix trick due to A. Connes (see [2]). We quickly review basic facts on the former. For a unital C^*-algebra A, we let $GL_n(A)$ the group of the invertible elements in the $n \times n$ matrices $M_n(A)$ over A (equipped with the C^*-norm). The inductive limit of topological groups $(\, GL_n(A) \,|\, n \in \mathbf{N}\,)$ with the usual embeddings $GL_n(A) \hookrightarrow GL_{n+1}(A)$ is denoted by $GL_\infty(A)$ and the connected component of the identity by $GL^0_\infty(A)$. Suppose that τ is a tracial state on A. If ξ is a piecewise continuously differentiable mapping from $[0, 1]$ into $GL^0_\infty(A)$, we define

$$\widetilde{\Delta}_\tau(\xi) = \frac{1}{2\pi i}\int_0^1 \tau\left(\dot{\xi}(t)\xi(t)^{-1}\right) dt$$

(note that the range of ξ is contained in $GL_n(A)$ for some n since [0,1] is compact and that τ actually means $\tau \otimes \mathrm{tr}$ on $A \otimes M_n = M_n(A)$). The determinant Δ_τ ([14]) associated with a tracial state τ is the mapping from $GL^0_\infty(A)$ into $\mathbf{C}/\tau_*(K_0(A))$ defined by

$$\Delta_\tau(x) = p(\widetilde{\Delta}_\tau(\xi)) \ .$$

Here p is the quotient mapping from $\mathbf{C}$ onto $\mathbf{C}/\tau_*(K_0(A))$, and ξ is a piecewise continuously differentiable mapping from [0,1] into $GL^0_\infty(A)$ with $\xi(0) = 1$ and $\xi(1) = x$.

A crucial fact here is that Δ_τ is a group homomorphism. For a unitary $x \in A$ with $\|x - 1\| < 1$, the logarithm $h = i^{-1}\log(x)$ (with the principal branch) makes sense and we can consider the path $\xi(t) = \exp(iht)$ from 1 to x. Since $\dot{\xi}(t)\xi(t)^{-1} = ih$, we have

$$\Delta_\tau(x) = \frac{1}{2\pi i} p\left(\int_0^1 \tau(ih)dt\right) = \frac{1}{2\pi i} p(\tau(\log(x))) \ .$$

Proof of Proposition 8

Let α and β be approximately conjugate $\mathbf{Z}^N$-actions on A. In general, if an automorphism of a simple unital C^*-algebra is close to the identity in norm then it is inner, and furthermore it is implemented by a unitary which is also close to the unit of the algebra. Hence for a sufficiently small $\varepsilon > 0$ there exist unitaries $u_1, \ldots, u_N$ in A and an automorphism γ of A such that

$$Ad\, u_i \circ \alpha_{\xi_i} = \gamma \circ \beta_{\xi_i} \circ \gamma^{-1} \ ,$$

$$\|u_i - 1\| < \varepsilon$$

for $i = 1, \ldots, N$. We want to show

$$u_k \alpha_{\xi_k}(u_l) = u_l \alpha_{\xi_l}(u_k) \tag{12}$$

for any $k, l = 1, \ldots, N$. From the commutativity of α_{ξ_k}, and the simplicity of A, there exists $\lambda \in \mathbf{T}$ such that

$$(u_l \alpha_{\xi_l}(u_k))^* u_k \alpha_{\xi_k}(u_l) = \lambda 1 \ .$$

Since u_k, u_l are close to 1, we can set

$$h_k = \frac{1}{2\pi i} \log(u_k), \quad h_l = \frac{1}{2\pi i} \log(u_l)$$

and

$$H(s) = \frac{1}{2\pi i} \log \{(u_l^s \alpha_{\xi_l}(u_k^s))^* u_k^s \alpha_{\xi_k}(u_l^s)\}$$

for $s \in [0, 1]$. Applying Δ_τ to the both sides of the equality

$$e^{-2\pi i \alpha_{\xi_l}(sh_k)} e^{-2\pi i s h_l} e^{2\pi i s h_k} e^{2\pi i \alpha_{\xi_k}(sh_l)} = e^{2\pi i H(s)} \ ,$$

we have

$$-p(\tau(\alpha_{\xi_l}(sh_k))) - p(\tau(sh_l)) + p(\tau(sh_k)) + p(\tau(\alpha_{\xi_k}(sh_l))) = p(\tau(H(s))) \ .$$

The uniqueness of a trace shows that the left hand side of the above equality is zero, i.e., $\tau(H(s)) \in \tau_*(K_0(A))$ for any $s \in [0, 1]$. Since $\tau_*(K_0(A))$ is discrete in $\mathbf{C}$ and $\tau(H(0)) = 0$, we obtain $H(1) = \tau(H(1)) = 0$ i.e., $\lambda = 1$. Using these unitaries $u_1, \ldots, u_N$, we construct a desired 1-cocycle by the method in [2]. We consider the $\mathbf{Z}^N$-action σ on $M_2(A)$ defined by

$$\sigma_{\xi_i} = Ad \begin{bmatrix} 1 & 0 \\ 0 & u_i \end{bmatrix} \circ \alpha_{\xi_i} \ .$$

Since $\sigma_{\xi_1}, \ldots, \sigma_{\xi_N}$ commute with each others from (12), σ is indeed well-defined. Note that

$$\sigma_g\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} \alpha_g(x) & 0 \\ 0 & \gamma \circ \beta_g \circ \gamma^{-1}(y) \end{bmatrix}$$

for any $g \in \mathbf{Z}^N$ and $x, y \in A$. The identity

$$\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

shows

$$\gamma \circ \beta_g \circ \gamma^{-1}(x) = Ad\, u_g \circ \alpha_g(x) \ ,$$

where u_g is the desired 1-cocycle defined by

$$\begin{bmatrix} 0 & 0 \\ u_g & 0 \end{bmatrix} = \sigma_g\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$

(see [22, Lemma 8.11.2]) . ∎

Remark 9 If α and β are outer conjugate automorphisms of a UHF algebra with the Rohlin property, then they are approximately conjugate by the stability property. Hence the three notions of conjugacy defined above are the same for those automorphisms. However outer conjugacy does not imply approximate conjugacy for $\mathbf{Z}^N$-actions. See Remark 18 for a counter-example.

4 Product type actions

In this section we discuss product type $\mathbf{Z}^2$-actions on UHF algebras. As in the case of single automorphisms, the Rohlin property for these actions is closely related to a notion of uniform distribution of points in $\mathbf{T}^2$. First we say the N-dimensional version of [1, Lemma 4.1]. This can be shown as in the one-dimensional case, so we omit the proof.

Proposition 10 *Let $(\, S_k \,|\, k \in \mathbf{N} \,)$ be a sequence of finite sequences in $\mathbf{T}^N$ i.e.,*

$$S_k = (\, s_{k,p} \,|\, p = 1, \ldots, n_k \,) \;,$$
$$s_{k,p} \in \mathbf{T}^N$$

for each $k \in \mathbf{N}$ and $p = 1, \ldots, n_k$. Then the following conditions on $(\, S_k \,|\, k \in \mathbf{N} \,)$ are equivalent.

(1)

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{p=1}^{n_k} f(s_{k,p}) = \int_{\mathbf{T}^n} f(s) ds$$

for any $f \in C(\mathbf{T}^N)$, where ds denotes the normalized Haar measure on $\mathbf{T}^N$.

(2)

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{p=1}^{n_k} s_{k,p}^l = 0$$

for any $l = (l_1, \ldots, l_N) \in \mathbf{Z}^N \setminus \{0\}$, where s^l denotes $s_1^{l_1} \cdots s_N^{l_N}$ for each $s = (s_1, \ldots, s_N) \in \mathbf{T}^N$.

(3)

$$\lim_{k\to\infty} \frac{1}{n_k} \nu_k \left(\prod_{i=1}^{N} [\, \theta_1^{(i)}, \theta_2^{(i)}) \right) = (2\pi)^{-N} \prod_{i=1}^{N} (\theta_2^{(i)} - \theta_1^{(i)})$$

for any $0 \le \theta_1^{(i)} \le \theta_2^{(i)} < 2\pi$, where ν_k is defined by

$$\nu_k(S) = \sharp\{\; p \mid 1 \le p \le n_k \text{ and } \arg(s_{k,p}) \in S \;\}$$

for each subset S of $\prod_{i=1}^{N} [\, 0, 2\pi)$ and $\sharp F$ denotes the cardinality of the set F.

These conditions necessarily imply that $n_k \to \infty$. Moreover suppose that $(\, n_k \,|\, k \in \mathbf{N}\,)$ has the following asymptotic factorization into large factors: For any $n \in \mathbf{N}$ there exists positive integer k_0 such that for any positive integer $k \geq k_0$ one has $n_k^{(1)}, \ldots, n_k^{(N)} \geq n$ where $n_k^{(i)}$ are the components of n_k, i.e., $n_k = n_k^{(1)} \cdots n_k^{(N)}$. Then the above conditions are also equivalent to

(4) *For any $\varepsilon > 0$ there exist positive integers k_0 and n_0 such that for any k, $n_k^{(1)}, \ldots, n_k^{(N)} \in \mathbf{N}$ satisfying $k \geq k_0$, $n_k^{(1)}, \ldots, n_k^{(N)} \geq n_0$ and $n_k = n_k^{(1)} \cdots n_k^{(N)}$, there exists an bijection φ from $\{1, \ldots, n_k\}$ onto $\{1, \ldots, n_k^{(1)}\} \times \cdots \times \{1, \cdots, n_k^{(N)}\}$ such that*

$$\left| s_{k,p} - (\exp\left(2\pi i \cdot \frac{(\varphi(p))_1}{n_k^{(1)}}\right), \ldots, \exp\left(2\pi i \cdot \frac{(\varphi(p))_N}{n_k^{(N)}}\right)) \right| < \varepsilon \qquad (13)$$

for any k and p, where $|s| \equiv \max\{\, |s_p| \; : 1 \leq p \leq N\,\}$ for each $s \in \mathbf{T}^N$ and $(\varphi(p))_i$ denotes the i-th component of $\varphi(p)$.

If S_k satisfies the estimate (13) for some φ as above, then S_k is said to be $(n_k^{(1)}, \ldots, n_k^{(N)}; \varepsilon)$ *-distributed*. If one of the conditions of the above proposition holds then $(\, S_k \,|\, k \in \mathbf{N}\,)$ is said to be *uniformly distributed*.

Definition 11 Let α be a $\mathbf{Z}^N$-action on a UHF algebra A. Then α is said to be a *product type action* if there exists a sequence $(\, m_k \,|\, k \in \mathbf{N}\,)$ of positive integers such that $A \cong \otimes_{k=1}^{\infty} M_{m_k}$ and

$$\alpha_g(A_k) = A_k$$

for any $g \in \mathbf{Z}^N$ and $k \in \mathbf{N}$, where A_k denotes the C^*-subalgebra of A corresponding to $M_{m_k} \otimes (\otimes_{l \neq k} \mathbf{C} 1_{m_l})$.

Remark 12 In the situation above, if $N = 2$, then one finds unitaries $u_k^{(1)}, u_k^{(2)}$ in A_k and $\lambda_k \in \mathbf{T}$ such that

$$\alpha_{(p,q)} \lceil A_k = Ad\, {u_k^{(1)}}^p {u_k^{(2)}}^q \;,$$
$$u_k^{(1)} u_k^{(2)} = \lambda_k u_k^{(2)} u_k^{(1)}$$

for any $p, q \in \mathbf{Z}$. Since $u_k^{(1)}, u_k^{(2)}$ are unique up to a constant multiple, λ_k is unique. In addition $\lambda_k^{m_k} = 1$. For if μ_1 is an eigenvalue of $u_k^{(2)}$ with multiplicity r_1 then $\mu_1 \lambda_k^{-p}$ is also an eigenvalue of $u_k^{(2)}$ with multiplicity r_1 for each $p \in \mathbf{N}$. Since M_{m_k} is finite-dimensional, there exists $p_0 \in \mathbf{N}$ such that $\lambda_k^{p_0} = 1$ and $\lambda_k^p \neq 1$ for any $p = 1, \ldots, p_0 - 1$. If $\{\, \mu_1 \lambda_k^{-p} \,|\, p = 0, \ldots, p_0 - 1\,\}$ does not exhaust all the eigenvalues of $u_k^{(2)}$ then we take an eigenvalue μ_2 of $u_k^{(2)}$ not belonging to $\{\, \mu_1 \lambda_k^{-p} \,|\, p = 0, \ldots, p_0 - 1\,\}$ and repeat the same process. Thus there exist eigenvalues $\mu_1, \ldots, \mu_s$ of $u_k^{(2)}$ with multiplicity $r_1, \ldots, r_s$ respectively. Since $m_k = (r_1 + \cdots + r_s) p_0$, it follows that $\lambda_k^{m_k} = 1$.

For $n \times n$ unitary matrices U and V with $UV = VU$, we define $\mathrm{Sp}(U)$ to be a sequence consisting of the eigenvalues of U, each repeated as often as multiplicity dictates and $\mathrm{Sp}(U, V)$ is a sequence consisting of the pairs of eigenvalues of U and V with a common eigenvector, each repeated as often as multiplicity dictates. Then the Rohlin property for the product type $\mathbf{Z}^2$-actions on A with $\lambda_k = 1$ can be characterized as follows.

Proposition 13 *Let α be a product type $\mathbf{Z}^2$-action on a UHF algebra A with $(m_k \mid k \in \mathbf{N})$, $(u_k^{(1)} \mid k \in \mathbf{N})$, $(u_k^{(2)} \mid k \in \mathbf{N})$, $(\lambda_k \mid k \in \mathbf{N})$ as above. If $\lambda_k = 1$ for each $k \in \mathbf{N}$ then the following conditions are equivalent:*

(1) *α has the Rohlin property.*

(2) $(\,\mathrm{Sp}(\otimes_{k=m}^n u_k^{(1)}, \otimes_{k=m}^n u_k^{(2)}) \mid n = m,\ m{+}1,\ \ldots\,)$ *is uniformly distributed for any $m \in \mathbf{N}$.*

Proof. By Corollary 4, (1) is equivalent to the condition: $\alpha_{\xi_1}^p \alpha_{\xi_2}^q$ has the Rohlin property as a single automorphism of A for each $(p, q) \in \mathbf{Z}^2 \setminus \{0\}$. By [16, Lemma 5.2] this condition is equivalent to the condition:

$$(\ \mathrm{Sp}\left(\bigotimes_{k=m}^{n} {u_k^{(1)}}^p {u_k^{(2)}}^q\right) \ \mid n = m,\ m+1,\ \ldots\ )$$

is uniformly distributed in $\mathbf{T}$. By Proposition 10 for $N = 1$, the last condition is equivalent to the condition: for any $m \in \mathbf{N}$

$$\lim_{n\to\infty} \frac{1}{N(m,n)} \sum_{(\lambda_1,\lambda_2)\in \mathrm{Sp}\left(\otimes_{k=m}^n u_k^{(1)}, \otimes_{k=m}^n u_k^{(2)}\right)} \lambda_1^p \lambda_2^q = 0\ ,$$

where $N(m,n) \equiv \prod_{k=m}^n m_k$. Finally by Proposition 10 for $N = 2$, the last condition is equivalent to (2). ∎

In [16] A.Kishimoto showed the following for a UHF algebra A.

(1) Any two product type $\mathbf{Z}$-actions on A with the Rohlin property are approximately conjugate.

(2) For any $\mathbf{Z}$-action α on A with the Rohlin property and $\varepsilon > 0$, there exist a product type $\mathbf{Z}$-action β on A with the Rohlin property and an automorphism γ of A such that $\alpha \overset{\gamma,\varepsilon}{\approx} \beta$.

In particular there is one and only one approximate conjugacy class of $\mathbf{Z}$-actions on A with the Rohlin property. In the case of $N = 2$ we do not know whether (2) is valid or not. In the rest of this section we state several versions of (1) for $\mathbf{Z}^2$.

Theorem 14 *Let α and β be product type $\mathbf{Z}^2$-actions on a UHF algebra A with the Rohlin property. Let α be determined by $(\, m_k \,|\, k \in \mathbf{N}\,)$, $(\, \lambda_k \,|\, k \in \mathbf{N}\,)$ as in Definition 11 and Remark 12, and β by $(\, n_l \,|\, l \in \mathbf{N}\,)$, $(\, \mu_l \,|\, l \in \mathbf{N}\,)$. If $\lambda_k = \mu_l = 1$ for each $k, l \in \mathbf{N}$, then α and β are approximately conjugate.*

Proof. By patching several parts of the M_{m_k}'s and the M_{n_l}'s respectively there exist a sequence $(\, N_k \,|\, k \in \mathbf{N}\,)$ of integers satisfying $N_0 = 0$ and $N_k > 0$ $(k \geq 1)$ and sequences $(\, U_k^{(1)} \,|\, k \in \mathbf{N}\,)$, $(\, U_k^{(2)} \,|\, k \in \mathbf{N}\,)$ of unitary matrices such that

$$U_k^{(i)} \in U(M_{N_k} \otimes M_{N_{k+1}}) \;,$$

$$U_k^{(1)} U_k^{(2)} = U_k^{(2)} U_k^{(1)} \quad (i = 1, 2,\; k \in \mathbf{N}) \;,$$

$$(A, \alpha_{\xi_1}, \alpha_{\xi_2}) \cong \left(\bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} Ad\, U_{2k}^{(1)}, \bigotimes_{k=0}^{\infty} Ad\, U_{2k}^{(2)} \right) \;,$$

$$(A, \beta_{\xi_1}, \beta_{\xi_2}) \cong \left(\bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} Ad\, U_{2k+1}^{(1)}, \bigotimes_{k=0}^{\infty} Ad\, U_{2k+1}^{(2)} \right) \;.$$

Let $\varepsilon > 0$. Since $(\, \mathrm{Sp} \left(\otimes_{k=0}^{n} Ad\, U_{2k+1}^{(1)}, \otimes_{k=0}^{n} Ad\, U_{2k+1}^{(2)} \right) \,|\, n \in \mathbf{N}\,)$ is uniformly distributed by Proposition 13 and $U_0^{(1)} U_0^{(2)} = U_0^{(2)} U_0^{(1)}$, for a sufficiently large n there exist a unitary W_1 in $\otimes_{k=0}^{2n+2} M_{N_k}$ and unitaries $V_2^{(1)}, V_2^{(2)}$ in $\otimes_{k=2}^{2n+2} M_{N_k}$ such that

$$\| W_1 \left(\bigotimes_{k=0}^{n} U_{2k+1}^{(1)} \right) W_1^* - U_0^{(1)} \otimes V_2^{(1)} \| < 2^{-2} \varepsilon \;,$$

$$\| W_1 \left(\bigotimes_{k=0}^{n} U_{2k+1}^{(2)} \right) W_1^* - U_0^{(2)} \otimes V_2^{(2)} \| < 2^{-2} \varepsilon \;.$$

Then replace M_{N_2} by $\bigotimes_{k=2}^{2n+2} M_{N_k}$ and $U_1^{(i)}$ by $\bigotimes_{k=0}^{n} U_{2k+1}^{(i)}$, $U_2^{(i)}$ by $\bigotimes_{k=1}^{n+1} U_{2k}^{(i)}$ $(i = 1, 2)$ respectively, and further replace $M_{N_{k-2n}}$ by M_{N_k} $(k \geq 2n+3)$ and $U_{2(k-n)+1}^{(i)}$ by $U_{2k+1}^{(i)}$, $U_{2(k-n)+2}^{(i)}$ by $U_{2k+2}^{(i)}$ $(i = 1, 2,\; k \geq n+1)$ respectively. Thereby $W_1 \in U(M_{N_1} \otimes M_{N_2})$ and $V_2^{(1)}, V_2^{(2)} \in U(M_{N_2})$, which satisfy

$$\| Ad\, W_1(U_1^{(1)}) - U_0^{(1)} \otimes V_2^{(1)} \| < 2^{-2} \varepsilon \;,$$

$$\| Ad\, W_1(U_1^{(2)}) - U_0^{(2)} \otimes V_2^{(2)} \| < 2^{-2} \varepsilon \;.$$

In the same way, after replacing M_{N_3}, $U_2^{(i)}$, $U_3^{(i)}$ etc. suitably, there exist a unitary W_2 in $M_{N_2} \otimes M_{N_3}$ and unitaries $V_3^{(1)}, V_3^{(2)}$ in M_{N_3} such that

$$\| Ad\, W_2(U_2^{(1)}) - V_2^{(1)} \otimes V_3^{(1)} \| < 2^{-3} \varepsilon \;,$$

$$\|Ad\,W_2(U_2^{(2)}) - V_2^{(2)} \otimes V_3^{(2)}\| < 2^{-3}\varepsilon\ .$$

By repeating the above procedure for $k = 3, 4, \ldots$, we can construct a unitary W_k in $M_{N_k} \otimes M_{N_{k+1}}$ and unitaries $V_{k+1}^{(1)}, V_{k+1}^{(2)}$ in $M_{N_{k+1}}$ in such a way that

$$\|Ad\,W_k(U_k^{(1)}) - V_k^{(1)} \otimes V_{k+1}^{(1)}\| < 2^{-(k+1)}\varepsilon\ ,$$

$$\|Ad\,W_k(U_k^{(2)}) - V_k^{(2)} \otimes V_{k+1}^{(2)}\| < 2^{-(k+1)}\varepsilon\ .$$

Thus

$$\begin{aligned}(A, \alpha_{\xi_1}, \alpha_{\xi_2}) \quad &\cong \quad \left(\bigotimes_{k=0}^{\infty} M_{N_k},\ \bigotimes_{k=0}^{\infty} Ad\,U_{2k}^{(1)},\ \bigotimes_{k=0}^{\infty} Ad\,U_{2k}^{(2)} \right) \\ &\overset{\gamma_e, \frac{1}{3}\varepsilon}{\approx} \quad \left(\bigotimes_{k=0}^{\infty} M_{N_k},\ \bigotimes_{k=0}^{\infty} Ad\,V_k^{(1)},\ \bigotimes_{k=0}^{\infty} Ad\,V_k^{(2)} \right)\ ,\end{aligned}$$

where $\gamma_e \equiv \otimes_{k=1}^{\infty} Ad\,W_{2k}$ and

$$\begin{aligned}(A, \beta_{\xi_1}, \beta_{\xi_2}) \quad &\cong \quad \left(\bigotimes_{k=0}^{\infty} M_{N_k},\ \bigotimes_{k=0}^{\infty} Ad\,U_{2k+1}^{(1)},\ \bigotimes_{k=0}^{\infty} Ad\,U_{2k+1}^{(2)} \right) \\ &\overset{\gamma_o, \frac{2}{3}\varepsilon}{\approx} \quad \left(\bigotimes_{k=0}^{\infty} M_{N_k},\ \bigotimes_{k=0}^{\infty} Ad\,V_k^{(1)},\ \bigotimes_{k=0}^{\infty} Ad\,V_k^{(2)} \right)\ ,\end{aligned}$$

where $\gamma_o \equiv \otimes_{k=0}^{\infty} Ad\,W_{2k+1}$. This completes the proof. ∎

As mentioned in the introduction, we discuss product type actions for two classes of UHF algebras. Let $(\, p_k \,|\, k \in \mathbf{N}\,)$ be the prime numbers in the increasing order. For a sequence $(\, i_k \,|\, k \in \mathbf{N}\,)$ of nonnegative integers with $\sum_{k=1}^{\infty} i_k = \infty$, put $q_k = p_k^{i_k}$ and let $A = \otimes_{k=1}^{\infty} M_{q_k}$. We regard M_{q_k} as a C^*-subalgebra of A. We consider the class of product type $\mathbf{Z}^2$-actions α on this A. Assume that α looks like

$$\alpha_{(p,q)} \lceil M_{q_k} = Ad\, {u_k^{(1)}}^p {u_k^{(2)}}^q \tag{14}$$

on M_{q_k} with unitaries $u_k^{(1)}, u_k^{(2)}$ in M_{q_k} and $\lambda_k \in \mathbf{T}$ satisfying $u_k^{(1)} u_k^{(2)} = \lambda_k u_k^{(2)} u_k^{(1)}$. Since $\lambda_k^{q_k} = 1$, we may regard λ_k as an element of $G_k \equiv \mathbf{Z}/q_k\mathbf{Z}$. We let $[\alpha]$ be the sequence $(\, \lambda_k \,|\, k \in \mathbf{N}\,)$ in $\prod_{k=1}^{\infty} G_k$. We define an equivalence relation in $\prod_{k=1}^{\infty} G_k$ by: $g \sim h$ if there is an n such that $g_k = h_k$ for all $k \geq n$. Let 0 be the trivial sequence $(\,0, 0, \ldots)$. We note that for every $g \in \prod_{k=1}^{\infty} G_k$ there is an action α in the above class with $[\alpha] = g$.

Theorem 15 (1) *If α is an action in the above class and $[\alpha] \not\sim 0$, then α has the Rohlin property.*
(2) *If α and β are actions in the above class and satisfy the Rohlin property, then the following are equivalent:*

(2.1) $[\alpha] \sim [\beta]$.

(2.2) *α and β are outer conjugate.*

Before proving Theorem 15 we introduce some notations and prepare a lemma. For a positive integer n and $\lambda \in \mathbf{T}$ with $\lambda^n = 1$, we define the $n \times n$ unitary matrices $S(n)$ and $\Omega(n, \lambda)$ by

$$S(n) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}, \quad \Omega(n, \lambda) = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda^{n-1} \end{bmatrix}.$$

Lemma 16 *Suppose that U and V are $n \times n$ unitary matrices such that*

$$UV = \exp(2\pi ik/n)\, VU$$

for some $k \in \mathbf{N}$. Let q/p be the irreducible form of k/n. Then there exist $\omega_i, \mu_i \in \mathbf{T}$ $(i = 1, \ldots, n/p)$ such that (U, V) is conjugate to $(U_1 \otimes U_2, V_1 \otimes V_2)$, where

$$\lambda = \exp(2\pi ik/n)$$
$$U_1 = S(p)\ , \quad V_1 = \Omega(p, \lambda^{-1})\ ,$$
$$U_2 = \bigoplus_{i=1}^{n/p} \omega_i\ , \quad V_2 = \bigoplus_{i=1}^{n/p} \mu_i\ .$$

Moreover each ω_i, μ_i are unique up to multiples of powers of λ.

Proof. Since $U^p V = V U^p$ we have a complete orthonomal system of $\mathbf{C}^n$ consisting of the common eigenvectors of U^p and V. We take such a system $(\xi_{(\kappa,\mu)} | \kappa, \mu)$ i.e.,

$$U^p \xi_{(\kappa,\mu)} = \kappa \xi_{(\kappa,\mu)}\ ,$$
$$V \xi_{(\kappa,\mu)} = \mu \xi_{(\kappa,\mu)}.$$

Then if $\omega^p = \kappa$, the space spanned by

$$\xi_{(\kappa,\mu)}, \quad \omega U \xi_{(\kappa,\mu)}, \quad \ldots, \quad \omega^{p-1} U^{p-1} \xi_{(\kappa,\mu)}$$

is invariant under U, V, and the matrix representation of (U, V) with respect to the above basis is $(\omega U_1, \mu V_1)$. Thus (U, V) is conjugate to the direct sum of $(\omega_i U_1, \mu_i V_1)$ for some sequences ω_i, μ_i in $\mathbf{T}$. Since $(\omega U_1, \mu V_1)$ is conjugate to $(\omega \lambda^k U_1, \mu \lambda^j V_1)$ for all k, j, the last statement is obvious. ∎

Proof of Theorem 15 (1)

Let α be given as in the theorem. Take unitaries $u_k^{(1)}, u_k^{(2)}$ in M_{q_k} and $\lambda_k \in \mathbf{T}$ as in (14). By Corollary 4 it suffices to prove that $\alpha_{(p,q)}$ has the Rohlin property as a single automorphism for each $(p,q) \in \mathbf{Z}^2 \setminus \{0\}$. From the assumption there is a subsequence $(\, p_{k_n} \,|\, n \in \mathbf{N}\,)$ of $(\, p_k \,|\, k \in \mathbf{N}\,)$ such that $\lambda_{k_n} \neq 1$ for any n. Applying Lemma 16 to $u_{k_n}^{(1)}, u_{k_n}^{(2)}$ we have a decomposition $(u_{k_n,1}^{(1)} \otimes u_{k_n,2}^{(1)}, u_{k_n,1}^{(2)} \otimes u_{k_n,2}^{(2)})$ of $(u_{k_n}^{(1)}, u_{k_n}^{(2)})$ up to conjugacy, where

$$u_{k_n,1}^{(1)} \equiv S(p_{k_n}^{j_n}) \ , \quad u_{k_n,1}^{(2)} \equiv \Omega(p_{k_n}^{j_n}, \lambda_{k_n}^{-1})$$

for some $1 \leq j_n \leq i_{k_n}$. Then it is easy to see that

$$(\, \mathrm{Sp}\left({u_{k_n,1}^{(1)}}^{p} {u_{k_n,2}^{(2)}}^{q} \right) \,|\, n \in \mathbf{N}\,)$$

is uniformly distributed in $\mathbf{T}$. This ensures that

$$(\, \mathrm{Sp}\left(\bigotimes_{k=k_0}^{l} {u_k^{(1)}}^p {u_k^{(2)}}^q \right) \,|\, l \geq k_0 \,)$$

is also uniformly distributed for all k_0. Hence $\alpha_{(p,q)} = \otimes_{k=1}^{\infty} Ad\, {u_k^{(1)}}^p {u_k^{(2)}}^q$ has the Rohlin property by [16, Lemma 5.2]. ∎

To prove Theorem 15 (2) we introduce an invariant as a slight generalization of that in [9].

Definition 17 For $n \times n$ unitary matrices U, V and $\lambda \in \mathbf{T}$ with $\lambda^n = 1$, if $\|\lambda UV - VU\| < 2$ then the closed complex path $\gamma(t) = \det((1-t)\lambda UV + tVU)$ $(t \in [0,1])$ does not go through zero. We define $\omega_\lambda(U,V)$ as the winding number of the path γ around zero.

From $\|\lambda UV - VU\| < 2$ we can define $\log(\lambda^{-1} VUV^*U^*)$, with log the principal branch of the logarithm. As is shown in [10, Lemma 3.1]

$$\omega_\lambda(U,V) = \frac{1}{2\pi i} \mathrm{Tr}(\log(\lambda^{-1} VUV^*U^*))$$

with the nonnormalized trace Tr on M_n.

Proof of Theorem 15 (2).

Take $u_k^{(1)}, u_k^{(2)} \in U(M_{q_k})$, $\lambda_k \in \mathbf{T}$ as in (14) for α, and $v_k^{(1)}, v_k^{(2)} \in U(M_{q_k})$, $\mu_k \in \mathbf{T}$ for β similarly, i.e.,

$$\beta_{(p,q)} \lceil M_{q_k} = Ad\, {v_k^{(1)}}^p {v_k^{(2)}}^q,$$

$$v_k^{(1)} v_k^{(2)} = \mu_k v_k^{(2)} v_k^{(1)} \ .$$

First we show that (2.2) implies (2.1). To get a contradiction we assume that outer conjugate α, β satisfy $[\alpha] \not\sim [\beta]$. Outer conjugacy means

$$Ad\, W_i \circ \alpha_{\xi_i} = \gamma^{-1} \circ \beta_{\xi_i} \circ \gamma \quad (i = 1, 2)$$

for some unitaries $W_1, W_2 \in A$ and an automorphism γ of A. For any $\varepsilon > 0$ we have a positive integer M and unitaries W_1', W_2' in $\otimes_{k=1}^{M} M_{q_k}$ with $\|W_i - W_i'\| < \varepsilon$ $(i = 1, 2)$ so that

$$\|Ad\, W_i' \circ \alpha_{\xi_i} - \gamma^{-1} \circ \beta_{\xi_i} \circ \gamma\| < 2\varepsilon \ .$$

By the assumption there exists a positive integer $K > M$ with $\lambda_K \neq \mu_K$. Further take a suffciently large $N > K$ such that

$$\gamma(M_{q_K}) \subseteq_\varepsilon M_{q_1} \otimes \cdots \otimes M_{q_N} \ ,$$

where $X \subseteq_\varepsilon Y$ means that for any $x \in X$ there is $y \in Y$ satisfying $\|x - y\| \leq \varepsilon \|x\|$. Here we use the perturbation theorem [5, Corollary 6.8], that is, for a sufficiently small $\varepsilon > 0$ we have a unitary w_1 in A such that

$$Ad\, w_1 \circ \gamma(M_{q_K}) \subseteq M_{q_1} \otimes \cdots \otimes M_{q_N}$$

and $\|w_1 - 1\| < 28\varepsilon^{\frac{1}{2}}$. Set $\gamma_1 = Ad\, w_1 \circ \gamma,\ B_1 = \gamma_1(M_{q_K})$ and $B_2 = M_{q_1} \otimes \cdots \otimes M_{q_N} \cap B_1'$. Then

$$M_{q_1} \otimes \cdots \otimes M_{q_N} = B_1 \otimes B_2 \ ,$$

$$\gamma_1 \circ Ad\, W_i' \circ \alpha_{\xi_i} \circ \gamma_1^{-1}(B_1) = B_1 \ ,$$

$$\gamma_1 \circ Ad\, W_i' \circ \alpha_{\xi_i} \circ \gamma_1^{-1} \lceil B_1 = Ad\, \gamma_1(u_K^{(i)})$$

for $i = 1, 2$. Since $\|Ad\, W_1' \circ \alpha_{\xi_1} - \gamma_1^{-1} \circ \beta_{\xi_1} \circ \gamma_1\| \leq C_1 \varepsilon^{\frac{1}{2}}$ for some positive constant C_1 independent of ε we have

$$\beta_{\xi_1}(B_1) \subseteq_{C_1 \varepsilon^{\frac{1}{2}}} B_1 \ .$$

Noting that $B_1,\ \beta_{\xi_1}(B_1) \subseteq M_{q_1} \otimes \cdots \otimes M_{q_N}$, we can use [5, Corollary 6.8] again. So we have a unitary w_2 in $M_{q_1} \otimes \cdots \otimes M_{q_N}$ such that

$$Ad\, w_2 \circ \beta_{\xi_1}(B_1) \subseteq B_1 \ ,$$

$$\|w_2 - 1\| < C_2 \varepsilon^{\frac{1}{4}}$$

for some positive constant C_2. As $M_{q_1} \otimes \cdots \otimes M_{q_N}$ is finite-dimensional and

$$Ad\, w_2 \circ \beta_{\xi_1}(B_1) = B_1 \ ,$$

$$Ad\, w_2 \circ \beta_{\xi_1}(M_{q_1} \otimes \cdots \otimes M_{q_N}) = M_{q_1} \otimes \cdots \otimes M_{q_N} \ ,$$

we have unitaries U_1 in B_1 and U_2 in B_2 such that

$$Ad\, w_2 \circ \beta_{\xi_1} \lceil B_1 = Ad\, U_1 \ ,$$

$$Ad\, w_2 \circ \beta_{\xi_1} \lceil B_2 = Ad\, U_2.$$

For these unitaries we have the following estimates:

$$\begin{aligned}
\|(Ad\,\gamma_1(u_K^{(1)}) - Ad\,U_1)\lceil B_1\| &= \|(\gamma_1 \circ Ad\,W_1' \circ \alpha_{\xi_1} \circ \gamma_1^{-1} - Ad\,w_2 \circ \beta_{\xi_1})\lceil B_1\| \\
&\leq \|\gamma_1 \circ Ad\,W_1' \circ \alpha_{\xi_1} \circ \gamma_1^{-1} - Ad\,w_2 \circ \beta_{\xi_1}\| \\
&\leq C_4 \varepsilon^{\frac{1}{4}},
\end{aligned}$$

$$\begin{aligned}
\|Ad(U_1 \otimes U_2) - \bigotimes_{k=1}^{N} Ad\,v_k^{(1)}\| &\leq \|Ad\,w_2 \circ \beta_{\xi_1} - \beta_{\xi_1}\| \\
&\leq C_4 \varepsilon^{\frac{1}{4}}
\end{aligned}$$

for some positive constant C_4. Thus we obtain scalars $\eta_1,\ \eta_2$ of $\mathbf{T}$ such that

$$\|\gamma_1(u_K^{(1)}) - \eta_1 U_1\| < 4C_4\varepsilon^{\frac{1}{4}}\ ,$$

$$\|U_1 \otimes U_2 - \eta_2 \bigotimes_{k=1}^{N} v_k^{(1)}\| \leq\ 4C_4\varepsilon^{\frac{1}{4}}.$$

Consequently we have

$$\|\gamma_1(u_K^{(1)}) \otimes U_2 - \eta \bigotimes_{k=1}^{N} v_k^{(1)}\| < C_5\varepsilon^{\frac{1}{4}}$$

for some $\eta \in \mathbf{T}$ and positive constant C_5. Similarly for the direction of ξ_2 we obtain a unitary V_2 in B_2 such that

$$\|\gamma_1(u_K^{(2)}) \otimes V_2 - \zeta \bigotimes_{k=1}^{N} v_k^{(2)}\| < C_6\varepsilon^{\frac{1}{4}}$$

for some $\zeta \in \mathbf{T}$ and positive constant C_6. To use an invariant in Definition 17, we set $\mu = \prod_{k=1}^{N} \mu_k$. Then there is a $\lambda \in \mathbf{T}$ such that $\lambda^{q_1 \cdots q_N} = 1$ and

$$|\lambda\mu - 1| + 2(C_5 + C_6)\varepsilon^{\frac{1}{4}} < 2\ .$$

Note that ω_λ is invariant under homotopy of unitaries for which ω_λ is defined. From the above estimates we have

$$\omega_\lambda(\gamma_1(u_K^{(1)}) \otimes U_2, \gamma_1(u_K^{(2)}) \otimes V_2) = \omega_\lambda(\eta \bigotimes_{k=1}^{N} v_k^{(1)}, \zeta \bigotimes_{k=1}^{N} v_k^{(2)})$$

for a sufficiently small $\varepsilon > 0$. We now evaluate the both sides to get a contradiction. Let

$$\lambda_k = \exp(2\pi i \cdot \frac{s_k}{q_k}),\ \mu_k = \exp(2\pi i \cdot \frac{t_k}{q_k}),$$

$$\lambda = \exp(2\pi i \cdot \frac{s}{q_1 \cdots q_N})$$

for some $s_k, t_k \in \{0, \ldots, q_k - 1\}$ and $s \in \{0, \ldots, (q_1 \cdots q_N - 1)\}$. Then

$$\begin{aligned}
&\lambda^{-1}(\gamma_1(u_K^{(2)}) \otimes V_2)(\gamma_1(u_K^{(1)}) \otimes U_2)(\gamma_1(u_K^{(2)}) \otimes V_2)^*(\gamma_1(u_K^{(1)}) \otimes U_2)^* \\
&= \quad 1_{q_K} \otimes \exp\left\{2\pi i \left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_K}{q_K}\right)\right\} V_2 U_2 V_2^* U_2^* \\
&= \quad \exp\left(2\pi i \left\{\bigoplus_{j=1}^{q_K} \left(\left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_K}{q_K}\right) 1_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N} + H_2\right)\right\}\right)
\end{aligned}$$

for some $H_2 \in M_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N}$ with $\mathrm{Tr}(H_2) \in \mathbf{Z}$. Thus

$$\begin{aligned}
&\omega_\lambda(\gamma_1(u_K^{(1)}) \otimes U_2, \gamma_1(u_K^{(2)}) \otimes V_2) \\
&= \quad \mathrm{Tr}\left\{\bigoplus_{j=1}^{q_K} \left(\left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_K}{q_K}\right) 1_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N} + H_2\right)\right\} \\
&= \quad -s - s_K q_1 \cdots q_{K-1} q_{K+1} \cdots q_N + q_K \mathrm{Tr}(H_2).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\omega_\lambda\left(\eta \bigotimes_{k=1}^N v_k^{(1)}, \zeta \bigotimes_{k=1}^N v_k^{(2)}\right) &= \left(\frac{-s}{q_1 \cdots q_N} + \sum_{k=1}^N \frac{-t_k}{q_k} + n\right)\left(\prod_{l=1}^N q_l\right) \\
&= -s - \sum_{k=1}^N t_k q_1 \cdots q_{k-1} q_{k+1} \cdots q_N + n q_1 \cdots q_N
\end{aligned}$$

for some $n \in \mathbf{Z}$. Therefore

$$(s_K - t_K) q_1 \cdots q_{K-1} q_{K+1} \cdots q_N = q_K \mathrm{Tr}(H_2) + \sum_{\substack{k=1 \\ k \neq K}}^N \frac{-t_k}{q_k} + n q_1 \cdots q_N \ .$$

Noting that $s_K - t_K \neq 0$ and $s_K - t_K$ is not divided by q_K, we have a contradiction.

Next we show that (2.1) implies (2.2). We assume $[\alpha] \sim [\beta]$. Then we may also assume that $\lambda_k = \mu_k$ for any k since inner perturbation does not change outer conjugacy classes. If there is a $k_0 \in \mathbf{N}$ such that $\lambda_k = 1$ for any $k \geq k_0$, then we have the result from Theorem 14. If there is no such k_0, we pick up all the k's with $\lambda_k \neq 1$ and make the subsequence $(\, p_{k_n} \,|\, n \in \mathbf{N}\,)$ of $(\, p_k \,|\, k \in \mathbf{N}\,)$. Let $\lambda_k = \exp(2\pi i s_k / q_k)$ as before. Then for any N

$$\prod_{k=1}^{k_N} \lambda_k = \exp\left(2\pi i \cdot \sum_{n=1}^N \frac{s_{k_n}}{q_{k_n}}\right) \ .$$

By noting that $s_{k_n} \neq 0$ and q_{k_n}'s are relatively prime to the each others, $\sum_{n=1}^{N} s_{k_n}/q_{k_n}$ equals to

$$\frac{S_N}{p_{k_1}^{j_1} \cdots p_{k_N}^{j_N}}$$

in the irreducible form for some positive integers $j_1, \ldots, j_N$ and S_N. Here we apply Lemma 16 to two pairs $(\otimes_{k=1}^{k_N} u_k^{(1)}, \otimes_{k=1}^{k_N} u_k^{(2)})$, $(\otimes_{k=1}^{k_N} v_k^{(1)}, \otimes_{k=1}^{k_N} v_k^{(2)})$ of unitaries. Then for any $\varepsilon > 0$, if we take a sufficiently large N, these pairs are almost conjugate i.e., there is a unitary w_1 in $\otimes_{k=1}^{k_N} M_{q_k}$ such that

$$\| Ad\, w_1 \left(\bigotimes_{k=1}^{k_N} u_k^{(i)} \right) - \bigotimes_{k=1}^{k_N} v_k^{(i)} \| < 2^{-1}\varepsilon$$

for $i = 1, 2$. We adopt the same method for $\otimes_{k=k_N+1}^{\infty} M_{q_k}$ and $2^{-1}\varepsilon$ in place of ε. Repeating this procedure as in the proof of Theorem 14, we have the result. ∎

Remark 18 Let $A = M_3 \otimes M_{2^\infty}$ and let $\omega_n = \exp(2\pi i/n)$ for each $n \in \mathbf{N}$. We define $\mathbf{Z}^2$-actions α, β on A by

$$\alpha_{\xi_1} = Ad\,\Omega(3, \omega_3) \otimes \left(\bigotimes_{k=1}^{\infty} Ad\,\Omega(2^k, \omega_{2^k}) \right) \;, \quad \alpha_{\xi_2} = Ad\,S(3) \otimes \left(\bigotimes_{k=1}^{\infty} Ad\,S(2^k) \right) \;,$$

$$\beta_{\xi_1} = id_{M_3} \otimes \left(\bigotimes_{k=1}^{\infty} Ad\,\Omega(2^k, \omega_{2^k}) \right) \;, \quad \beta_{\xi_1} = id_{M_3} \otimes \left(\bigotimes_{k=1}^{\infty} Ad\,S(2^k) \right).$$

Then the same arguments as in the first part of the above proof show that α, β have the Rohlin property and they are not approximately conjugarte. However they are clearly outer conjugate.

Let $\{ q_k \,|\, k \in K \}$ be a finite or infinite set of prime numbers. We next consider product type $\mathbf{Z}^2$-actions on the UHF algebra

$$\bigotimes_{k \in K} M_{q_k^\infty} \;,$$

where $M_{q_k^\infty}$ is understood as $\otimes_{n=1}^{\infty} M_{q_k}$.

Theorem 19 *For the above UHF algebra, any two product type* $\mathbf{Z}^2$*-actions with the Rohlin property are approximately conjugate.*

Proof. From Theorem 14 it is enough to prove that for any product type $\mathbf{Z}^2$-action α on A with the Rohlin property and $\varepsilon > 0$, there exist an automorphism γ of A and a product type action β with the Rohlin property such that $\alpha \overset{\gamma,\varepsilon}{\approx} \beta$

and β has the same form as in Theorem 14 i.e., there exist a sequence $(\, n_l \,|\, l \in \mathbf{N}\,)$ of positive integers and sequences $(\, v_l^{(1)} \,|\, l \in \mathbf{N}\,)$, $(\, v_l^{(2)} \,|\, l \in \mathbf{N}\,)$ of unitary matrices such that

$$v_l^{(1)},\ v_l^{(2)} \in M_{n_l}\ ,$$
$$v_l^{(1)} v_l^{(2)} = v_l^{(1)} v_l^{(2)}\ ,$$
$$(A, \beta_{\xi_1}, \beta_{\xi_2}) \cong \left(\bigotimes_{l=1}^{\infty} M_{n_l}, \bigotimes_{l=1}^{\infty} Ad\, v_l^{(1)}, \bigotimes_{l=1}^{\infty} Ad\, v_l^{(2)} \right)\ .$$

Let $(\, p_k \,|\, k \in \mathbf{N}\,)$ be the prime numbers in the increasing order. By definition we find a sequence $(\, N_k \,|\, k \in \mathbf{N}\,)$ of positive integers in such a way that each $k \in \mathbf{N}$ there are nonnegative integers $m_k^{(1)}, \ldots m_k^{(N_k)}$, unitaries u_k, v_k and $\lambda_k \in \mathbf{T}$ satisfying

$$(A, \alpha_{\xi_1}, \alpha_{\xi_2}) \cong \left(\bigotimes_{k=1}^{\infty} M_{p_1^{m_k^{(1)}} \cdots p_{N_k}^{m_k^{(N_k)}}}, \bigotimes_{k=1}^{\infty} Ad\, u_k, \bigotimes_{k=1}^{\infty} Ad\, v_k \right)\ ,$$

$$u_k,\ v_k \in U(M_{p_1^{m_k^{(1)}} \cdots p_{N_k}^{m_k^{(N_k)}}}),\quad u_k v_k = \lambda_k v_k u_k,$$

$$m_k^{(1)} + \cdots + m_k^{(N_k)} \neq 0\ .$$

Set $Q_k = p_1^{m_k^{(1)}} \cdots p_{N_k}^{m_k^{(N_k)}}$. By Remark 12, $\lambda_k^{Q_k} = 1$ for each k. If $\lambda_k = 1$ for any k, we are done, so we assume that $\lambda_k \neq 1$ for some $k \in \mathbf{N}$. Take $k_1 \equiv \min\{\, k \in \mathbf{N} \,|\, \lambda_k \neq 1 \,\}$. For $n \geq k_1 + 1$ we define

$$N(n) = \prod_{k=k_1+1}^{n} Q_k, \quad \lambda(n) = \prod_{k=k_1+1}^{n} \lambda_k$$

and relatively prime integers $M(n)$ and $K(n)$ by

$$\lambda(n) = \exp\left(2\pi i \cdot \frac{K(n)}{M(n)} \right)\ ,$$

$$M(n) \leq N(n), \quad K(n) \in \{0, \ldots, M(n) - 1\}.$$

If we set $m^{(i)}(n) = \max\{\, m_k^{(i)} \,|\, k_1 + 1 \leq k \leq n \,\}$ $(i = 1, \ldots, N_{k_1})$ then for each i the exponent of the factor p_i in $M(n)$ is less than or equal to $m^{(i)}(n)$. Hence taking a sufficiently large n, we can make the exponent of the factor p_i in $N(n)M(n)^{-1}$ as large as we like if $m^{(i)}(n) \neq 0$. In particular $N(n)M(n)^{-1}$ is divided by $Q_{k_1}^2$ and $N(n)$ is much larger than $M(n)$.

We want to show that for any $\delta > 0$, which is much smaller than ε, there exist positive integers $n,\ m_1,\ m_2$ and unitary matrices U_i, V_i $(i = 1, 2, 3)$ and W such that

$$n \geq k_1 + 1, \quad m_1, m_2 \geq \delta^{-1},$$

$$m_1 m_2 = N(n)M(n)^{-1}Q_{k_1}^{-2},$$
$$U_1,\ V_1 \in M_{Q_{k_1}},\quad U_2,\ V_2 \in M_{M(n)Q_{k_1}},$$
$$U_3,\ V_3 \in M_{N(n)M(n)^{-1}Q_{k_1}^{-2}},\quad W \in M_{N(n)},$$
$$U_1V_1 = \lambda_{k_1}^{-1}V_1U_1,\quad U_3V_3 = V_3U_3,$$
$$\mathrm{Sp}(U_3, V_3)\ is\ (m_1, m_2; \delta) - distributed \tag{15}$$

and

$$\left(\bigotimes_{k=k_1+1}^{n} M_{Q_k},\ \bigotimes_{k=k_1+1}^{n} Ad\, u_k,\ \bigotimes_{k=k_1+1}^{n} Ad\, v_k \right) \overset{Ad\, W, 2^{-1}\varepsilon}{\approx}$$
$$(\ M_{Q_{k_1}} \otimes M_{M(n)Q_{k_1}} \otimes M_{N(n)M(n)^{-1}Q_{k_1}^{-2}},$$
$$Ad(U_1 \otimes U_2 \otimes U_3),\ Ad(V_1 \otimes V_2 \otimes V_3\ ))\ .$$

Suppose that we have shown this statement, then we can construct the required γ and β as follows. Set

$$n_1 = (Q_1 Q_2 \cdots Q_{k_1}) \cdot Q_{k_1} \cdot N(n)M(n)^{-1}Q_{k_1}^{-2}\ ,$$
$$v_1^{(1)} = \left(\bigotimes_{k=1}^{k_1} u_k \right) \otimes U_1 \otimes U_3\ ,$$
$$v_1^{(2)} = \left(\bigotimes_{k=1}^{k_1} v_k \right) \otimes V_1 \otimes V_3\ ,$$
$$W_1 = W\ .$$

Then $v_1^{(1)},\ v_1^{(2)} \in U(M_{n_1})$. Applying the same method to

$$\left(M_{M(n)Q_{k_1}} \otimes \left(\bigotimes_{k=n+1}^{\infty} M_{Q_k} \right),\ Ad\, U_2 \otimes \left(\bigotimes_{k=n+1}^{\infty} Ad\, u_k \right), \right.$$
$$\left. Ad\, V_2 \otimes \left(\bigotimes_{k=n+1}^{\infty} Ad\, v_k \right) \right)$$

and $2^{-1}\varepsilon$ in place of

$$\left(\bigotimes_{k=1}^{\infty} M_{Q_k},\ \bigotimes_{k=1}^{\infty} Ad\, u_k,\ \bigotimes_{k=1}^{\infty} Ad\, v_k \right)$$

and ε, we get $n_2, v_2^{(1)}, v_2^{(2)}, W_2$. Repeating this procedure as in the proof of Theorem 14, we obtain an automorphism γ as the infinite product of $Ad\,W_i$ $(i = 1, 2, \ldots)$ and an action β as

$$\beta_{\xi_i} = \bigotimes_{l=1}^{\infty} Ad\, v_l^{(i)} \quad (i = 1, 2) \ .$$

This β also has the Rohlin property due to (15). Hence we have the required γ and β.

Now we show the remaining part of the proof, that is, the existence of $n, m_1, m_2 \in \mathbf{N}$, unitary matrices U_i, V_i $(i = 1, 2, 3)$ and W satisfying the prescribed conditions. By Lemma 16 we can decompose (up to conjugacy)

$$\left(\bigotimes_{k=k_1+1}^{n} M_{Q_k}, \ \bigotimes_{k=k_1+1}^{n} u_k, \ \bigotimes_{k=k_1+1}^{n} v_k \right)$$

into

$$\left(M_{M(n)} \otimes M_{N(n)M(n)^{-1}}, U_1^{(n)} \otimes U_2^{(n)}, V_1^{(n)} \otimes V_2^{(n)} \right) \ ,$$

where $U_1^{(n)} = S(M(n))$, $V_1^{(n)} = \Omega(M(n), \lambda(n)^{-1})$ and $U_2^{(n)}, V_1^{(n)}$ are some commuting unitary matrices. Furthermore we see

$$\left(\mathrm{Sp}(U_2^{(n)}, V_2^{(n)}) \,|\, n \geq k_1 + 1 \right)$$

is uniformly distributed. Actually note that $\mathrm{Sp}(U_2^{(n)}, V_2^{(n)})$ is unique up to piecewise multiples of $(\lambda(n)^k, \lambda(n)^l)$ for any $k, l \in \mathbf{N}$. So if $\sup\{ M(n) \,|\, n \geq k_1 + 1 \} = \infty$ then it is clearly uniformly distributed. If $\sup\{ M(n) \,|\, n \geq k_1 + 1 \} < \infty$ then there is a positive integer M such that M is divided by $M(n)$ for any $n \geq k_1 + 1$. Noting that

$$\left(\bigotimes_{k=k_1+1}^{n} u_k \right) \left(\bigotimes_{k=k_1+1}^{n} v_k \right) = \exp\left(2\pi i \cdot \frac{K(n)}{M(n)} \right) \left(\bigotimes_{k=k_1+1}^{n} v_k \right) \left(\bigotimes_{k=k_1+1}^{n} u_k \right)$$

we have

$$\left(\bigotimes_{k=k_1+1}^{n} u_k^M \right) \left(\bigotimes_{k=k_1+1}^{n} v_k^M \right) = \left(\bigotimes_{k=k_1+1}^{n} v_k^M \right) \left(\bigotimes_{k=k_1+1}^{n} u_k^M \right)$$

for any $n \geq k_1 + 1$. This implies $u_k^M v_k^M = v_k^M u_k^M$ for any $k \geq k_1 + 1$. Hence by Proposition 13

$$\left(\ \mathrm{Sp} \left(\bigotimes_{k=k_1+1}^{n} u_k^M, \ \bigotimes_{k=k_1+1}^{n} v_k^M \right) \ \Big| \ n \geq k_1 + 1 \ \right)$$

is uniformly distributed. Since

$$\mathrm{Sp}\left(\bigotimes_{k=k_1+1}^{n} u_k^M,\ \bigotimes_{k=k_1+1}^{n} v_k^M\right) = \mathrm{Sp}\left(1\otimes {U_2^{(n)}}^M, 1\otimes {V_2^{(n)}}^M\right) \ ,$$

it follows that

$$(\ \mathrm{Sp}\left(U_2^{(n)},\ V_2^{(n)}\right)\ \mid n\ge k_1+1\ )$$

should be uniformly distributed. Using this distribution of the joint spectrum, we can make for a sufficiently large n

$$\left(M_{N(n)M(n)^{-1}}, U_2^{(n)}, V_2^{(n)}\right)$$

close to (up to conjugacy)

$$\left(M_{Q_{k_1}}\otimes M_{Q_{k_1}}\otimes M_{N(n)M(n)^{-1}Q_{k_1}^{-2}}, U_3'\otimes U_4'\otimes U_5^{(n)}, V_3'\otimes V_4'\otimes V_5^{(n)}\right)$$

in norm, where

$$U_3' = V_4' = S(Q_{k_1}),\ U_4' = V_3'^* = \Omega(Q_{k_1},\lambda_{k_1})$$

and $\mathrm{Sp}(U_5^{(n)}, V_5^{(n)})$ is $(m_1, m_2;\delta)$-distributed for some $m_1, m_2\in\mathbf{N}$ satisfying

$$m_1, m_2\ge\delta^{-1},\quad m_1m_2 = N(n)M(n)^{-1}Q_{k_1}^{-2}\ .$$

Since $U_4'V_4' = \lambda_{k_1}^{-1}V_4'U_4'$, we obtain desired unitaries in such a way that

$$U_1 = U_4'\ ,\quad V_1 = V_4'\ ,$$
$$U_2 = U_1^{(n)}\otimes U_3'\ ,\quad V_2 = V_1^{(n)}\otimes V_3'\ ,$$
$$U_3 = U_5^{(n)}\ ,\quad V_3 = V_5^{(n)}\ .$$

We complete the proof. ∎

Now we sum up the results we have shown so far. Let $(\,p_k\,|\,k\in\mathbf{N}\,)$ be the prime numbers in the increasing order. By Glimm's theorem ([11]), for any UHF algebra A there exists one and only one sequence $(\,i_k\,|\,k\in\mathbf{N}\,)$ of nonnegative integers or ∞ such that $A\cong\otimes_{k=1}^{\infty}M_{p_k^{i_k}}$, where $M_{p_k^{\infty}}$ is understood as $\otimes_{n=1}^{\infty}M_{p_k}$. Then our classification of $\mathbf{Z}^2$-actions on A is as follows:

Theorem 20 *Let A be a UHF algebra with the invariant $(\,i_k\,|\,k\in\mathbf{N}\,)$ as above.*
(1) *If $\sharp\{\,k\in\mathbf{N}\,|\,1\le i_k<\infty\,\}=\infty$ then there are infinitely many outer conjugacy classes of product type $\mathbf{Z}^2$-actions on A with the Rohlin property.*
(2) *If $\sharp\{\,k\in\mathbf{N}\,|\,1\le i_k<\infty\,\}<\infty$ and A is infinite-dimensional then there is one and only one outer conjugacy class of product type $\mathbf{Z}^2$-actions on A with the Rohlin property.*
(3) *If A is finite-dimensional then there is no $\mathbf{Z}^2$-action on A with the Rohlin property.*

Acknowledgements

The author would like to express his gratitude to Professor Akitaka Kishimoto for suggesting this line of the research and for helpful discussions. He also would like to thank the referees for the useful comments.

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